DGD 7

① Find $L = \lim\limits_{n\to\infty} 2^{\frac{n^2-1}{2n^2+2}}$

Solution $$\frac{n^2-1}{2n^2+2} = \frac{1-\overbrace{\frac{1}{n^2}}^{\to 0}}{2+\underbrace{\frac{2}{n^2}}_{\to 0}} \longrightarrow \frac{1}{2}$$

Therefore, by the continuity of the function $x \mapsto 2^x$,

$$L = 2^{1/2} = \sqrt{2}$$

② Find $L = \lim\limits_{n\to\infty}\left(1+\frac{2}{n}\right)^n$

Solution Use that $\lim\limits_{n\to\infty}\left(1+\frac{1}{n}\right)^n = e$,

then $$\left(1+\frac{2}{n}\right)^n = \left[\left(1+\frac{2}{n}\right)^{n/2}\right]^2 \longrightarrow e^2$$

③ Find $\lim\limits_{n\to\infty}\left(\sqrt{n^2+n}-n\right)$

Solution $$\sqrt{n^2+n}-n = \frac{\left(\sqrt{n^2+n}-n\right)\left(\sqrt{n^2+n}+n\right)}{\sqrt{n^2+n}+n}$$

$$= \frac{n}{\sqrt{n^2+n}+n} = \frac{1}{\underbrace{\sqrt{1+\frac{1}{n}}}_{\to 1}+1} \longrightarrow \frac{1}{2}$$

(4) Find $\lim_{n\to\infty} \frac{\sqrt{n^4-1}}{2n^2+n}$

<u>Solution</u>: $\frac{\sqrt{n^4-1}}{2n^2+n} = \frac{\overbrace{\sqrt{1-\frac{1}{n^4}}}^{\to 1}}{2+\underbrace{\frac{1}{n}}_{\to 0}} \longrightarrow \frac{1}{2}$

(5) Find $\lim_{n\to\infty} \left[\ln(2n^2+3) - 2\ln(n-1)\right]$

<u>Solution</u> $\ln(2n^2+3) - 2\ln(n-1)$

$= \ln \underbrace{\frac{2n^2+3}{(n-1)^2}}_{\to 2} \longrightarrow \ln 2$

(6) Express $1.2\overline{3}$ as a fraction.

<u>Solution</u>: $1.2\overline{3} = 1.2 + \frac{3}{10^2} + \frac{3}{10^3} + \cdots$

$= 1.2 + \frac{3}{10^2}\left(1 + \frac{1}{10} + \left(\frac{1}{10}\right)^2 + \cdots\right)$

$= 1.2 + \frac{3}{100} \cdot \frac{1}{1-\frac{1}{10}} = 1.2 + \frac{1}{30}$

$= \frac{12}{10} + \frac{1}{30} = \frac{37}{30}$

7) Find a formula for a_n if $a_0 = \frac{1}{2}$, $a_1 = -\frac{3}{8}$, $a_2 = \frac{9}{32}$, $a_3 = -\frac{27}{128}$, ... and find $\sum_{n=0}^{\infty} a_n$

Solution. Each consecutive term a_{n+1} is obtained from a_n by multiplying by $r = -\frac{3}{4}$, whence

$$a_n = a_0 \cdot \left(-\frac{3}{4}\right)^n = \frac{(-1)^n}{2} \cdot \frac{3^n}{4^n}, \text{ and}$$

$$\sum_{n=0}^{\infty} a_n = a_0 (1 + r + r^2 + \dots) = \frac{a_0}{1-r}$$

$$= \frac{1}{2} \cdot \frac{1}{1+\frac{3}{4}} = \frac{1}{2} \cdot \frac{1}{\frac{7}{4}} = \frac{1}{2} \cdot \frac{4}{7} = \frac{2}{7}$$

In Ex. 8–11 find whether the series is divergent, absolutely convergent or conditionally convergent

8) $\sum_{n=0}^{\infty} \frac{n^2 \cos n}{1+n^4}$

Solution: Use the comparison criterion:

$$|a_n| = \left| \frac{n^2 \cos n}{1+n^4} \right| \le \frac{n^2}{1+n^4} \le b_n = \frac{1}{n^2},$$

where $\sum b_n$ converges as the p-series with $p=2$

$\Longrightarrow$ the original series is abs. convergent

(9) $\sum_{n=0}^{\infty} (-1)^n \frac{2n}{3n+1}$

Solution. Since $\frac{2n}{3n+1} \to \frac{2}{3}$, $(-1)^n \frac{2n}{3n+1} \not\to 0 \Rightarrow$ the series diverges

(10) $\sum_{n=1}^{\infty} \frac{n+3}{n^2+1}$

Solution. Use the limit comparison criterion with $b_n = \frac{1}{n}$:

$$\frac{a_n}{b_n} = \frac{\frac{n+3}{n^2+1}}{\frac{1}{n}} = \frac{n(n+3)}{n^2+1} \to 1$$

Since $\sum b_n$ diverges, $\sum a_n$ also diverges

(11) $\sum_{n=1}^{\infty} \frac{(-1)^n \sin n \cdot (2^n-1)}{3^n-2}$

Solution. Use a combination of the comparison and root tests:

$$|a_n| = \frac{|\sin n| \cdot (2^n-1)}{3^n-2} \le \frac{2^n-1}{3^n-2} = b_n,$$

whereas $(b_n)^{1/n} \to \frac{2}{3} \Longrightarrow$

$\sum a_n$ is absolutely convergent

(12) How many entries of the series $\sum_{n=1}^{\infty} \frac{1}{n^4}$ should one take so that the error does not exceed 0.001?

Solution: The entries $a_n = \frac{1}{n^4}$ are non-negative and decreasing, whence $R_n \le \int_n^{\infty} \frac{dx}{x^4} = \frac{1}{3n^3}$, so that one has to find the minimal n with

$$\frac{1}{3n^3} \le 0.001, \text{ or } 3n^3 \geq 1000,$$

whence $n = 7$

(13) The infinite series $\sum_{n=1}^{\infty} (-1)^n \frac{n}{n^2+1}$ is replaced by the sum of the first 20 entries. Estimate the error.

Solution. The series is alternating, and $|a_n| = \frac{n}{n^2+1}$ is decreasing (take $f(x) = \frac{x}{x^2+1}$, then $f'(x) = [(x^2+1) - 2x^2]/(x^2+1)^2 = (1-x^2)/(x^2+1)^2 \le 0$ for $x \geq 1$). Therefore,

$$|R_n| \le |a_{n+1}| = \frac{21}{21^2+1} = \frac{21}{442}$$