0606

In problems 1–3 find whether the series is convergent, and if so, whether the convergence is absolute or conditional

(1) $\sum_{n=1}^{\infty} \frac{(-1)^n}{\sqrt{n}}$

Since $|a_n| = \frac{1}{\sqrt{n}} > 0$, the series is alternating; $|a_n| \to 0$ and monotone, so that the series converges

On the other hand, $\sum |a_n|$ diverges (as a series with polynomial entries $\sum \frac{1}{n^p}$, $p = \frac{1}{2} \leq 1$)

Therefore, $\sum a_n$ is conditionally convergent

(2) $\displaystyle\sum_{n=1}^{\infty} \frac{\sin n}{n^2}$

Let $a_n = \dfrac{\sin n}{n^2}$

Then $|a_n| = \dfrac{|\sin n|}{n^2} \le \dfrac{1}{n^2}$

Since $\sum \frac{1}{n^2}$ converges $(p = 2 > 1)$,

$\sum |a_n|$ also converges by

the comparison test $\Longrightarrow$

$\sum a_n$ is absolutely convergent

③ $\sum_{n=1}^{\infty} (-1)^n \frac{n}{n^2+4}$

Let $a_n = (-1)^n \frac{n}{n^2+4}$, so that

$$|a_n| = \frac{n}{n^2+4}$$

Then for $b_n = \frac{n}{n^2} = \frac{1}{n}$ (the ratio of the leading terms)

$$\lim \frac{|a_n|}{b_n} = 1$$

Since Σb_n diverges, $\Sigma |a_n|$ also diverges by the limit comparison test.

On the other hand, $|a_n| \to 0$ and $|a_n|$ is monotone (check: consider $f(x) = \frac{x}{x^2+4}$, $f'(x) = \frac{x^2+4-2x^2}{(x^2+4)^2} \leq 0$ for $x \geq 2$)

whence Σa_n is convergent by the alternating test

$\Rightarrow \Sigma a_n$ is conditionally convergent

In problems 4 – 7 use the ratio test to determine whether the series is convergent or divergent

(4) $\sum_{k=0}^{\infty} \frac{(-1)^k}{(2k)!}$

$a_k = \frac{(-1)^k}{(2k)!}$, $a_{k+1} = \frac{(-1)^{k+1}}{(2(k+1))!}$, whence

$\left| \frac{a_{k+1}}{a_k} \right| = \frac{(2k)!}{(2(k+1))!} = \frac{1 \cdot 2 \cdot \ldots \cdot 2k}{1 \cdot 2 \cdot \ldots (2k)(2k+1)(2k+2)}$

$= \frac{1}{(2k+1)(2k+2)} \xrightarrow[k \to \infty]{} 0$

Since $L = 0 < 1$, the series is absolutely convergent

5 $\displaystyle\sum_{n=1}^{\infty} \frac{(2n)!}{(n!)^2}$

Here $a_n = \dfrac{(2n)!}{n!\cdot n!}$, $a_{n+1} = \dfrac{(2n+2)!}{(n+1)!(n+1)!}$

$$\left|\frac{a_{n+1}}{a_n}\right| = \frac{(2n+2)!}{(n+1)!\,(n+1)!}\;\frac{n!\cdot n!}{(2n)!} =$$

$$= \frac{(2n+1)(2n+2)}{(n+1)^2} \longrightarrow 4$$

Therefore, $L = 4 > 1$,

whence the series is divergent

⑥ $$\sum_{n=1}^{\infty} (-1)^n \frac{2^n}{3^n \cdot n^3}$$

Here $$a_n = (-1)^n \frac{2^n}{3^n \cdot n^3}$$

$$a_{n+1} = (-1)^{n+1} \cdot \frac{2^{n+1}}{3^{n+1} \cdot (n+1)^3}$$

$$\left| \frac{a_{n+1}}{a_n} \right| = \frac{2^{n+1}}{3^{n+1} \cdot (n+1)^3} \cdot \frac{3^n \cdot n^3}{2^n}$$

$$= \frac{2}{3} \cdot \frac{n^3}{(n+1)^3} \longrightarrow \frac{2}{3}$$

Therefore, $L = \frac{2}{3} < 1$, so that the series is absolutely convergent

⑦ $$\sum_{n=1}^{\infty} \frac{n^{100}\ 100^{n}}{n!}$$

$$a_n = \frac{n^{100} \cdot 100^{n}}{n!}$$

$$a_{n+1} = \frac{(n+1)^{100} \cdot 100^{n+1}}{(n+1)!}$$

$$\left|\frac{a_{n+1}}{a_n}\right| = \frac{(n+1)^{100} \cdot 100^{n+1}}{(n+1)!} \cdot \frac{n!}{n^{100} \cdot 100^{n}}$$

$$= \underbrace{\left(\frac{n+1}{n}\right)^{100}}_{\to 1} \cdot \frac{100}{n+1} \longrightarrow 0$$

Therefore, $l = 0$, so $l < 1$

the series is convergent ✓

In problems (8) - (11) use the root test to determine whether the series is convergent or divergent

(8) $\displaystyle\sum_{n=2}^{\infty} \frac{(-1)^n}{(\ln n)^n}$

Here $|a_n| = \dfrac{1}{(\ln n)^n}$, so that

$$|a_n|^{1/n} = \frac{1}{\ln n} \longrightarrow 0,$$

so that $L = 0$, and

the series is absolutely convergent

9. $$\sum_{n=1}^{\infty} (-1)^n \frac{3^n+\sqrt{n}}{4^n-2^n}$$

$$|a_n| = \frac{3^n+\sqrt{n}}{4^n-2^n},$$

$$|a_n|^{1/n} = \frac{(3^n+\sqrt{n})^{1/n}}{(4^n-2^n)^{1/n}},$$ where

$$(3^n+\sqrt{n})^{1/n} = 3\Big(1+\underbrace{\frac{\sqrt{n}}{3^n}}_{\to 0}\Big)^{1/n} \to 3$$

$$(4^n-2^n)^{1/n} = 4\Big(1-\underbrace{\frac{2^n}{4^n}}_{\to 0}\Big)^{1/n} \to 4$$

Thus, $|a_n|^{1/n} \to L = \frac{3}{4}$

$\Rightarrow$ the series is absolutely convergent

10a) $\sum_{n=1}^{\infty} \left(1 + \frac{1}{n}\right)^{n^2}$

$$\left(a_n\right)^{1/n} = \left(\left(1 + \frac{1}{n}\right)^{n^2}\right)^{1/n}$$

$$= \left(1 + \frac{1}{n}\right)^{n} \longrightarrow e$$

Therefore, $L = e > 1$,

and the series is divergent

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$$\sum_{n=1}^{\infty}\left(\frac{-2n}{n+1}\right)^{5n}$$

$$|a_n|^{1/n} = \left(\left(\frac{2n}{n+1}\right)^{5n}\right)^{1/n}$$

$$= \left(\frac{2n}{n+1}\right)^{5} = \left(2\cdot\underbrace{\frac{n}{n+1}}_{\to 1}\right)^{5} \longrightarrow 2^5 = 32$$

Therefore, $L = 32 > 1$, so that the series diverges