ODE 5

In problems 1-5 one has to study the convergence of the given series

(1) $\sum_{n=1}^{\infty} \frac{n \sin^2 n}{1+n^3}$

<u>Solution</u> Use the comparison test

The original series is $\sum_{n=1}^{\infty} a_n$, $a_n = \frac{n \sin^2 n}{1+n^3}$,

so that $0 \le a_n \le \frac{n}{n^3} = \frac{1}{n^2}$,

i.e. $0 \le a_n \le b_n$ with $b_n = \frac{1}{n^2}$

The series $\sum_{n=1}^{\infty} b_n = \sum_{n=1}^{\infty} \frac{1}{n^2}$ converges

$\Longrightarrow \sum_{n=1}^{\infty} a_n$ also converges

2) $$\sum_{n=1}^{\infty} \frac{1+\cos n}{e^n}$$

<u>Solution</u> $\quad a_n = \frac{1+\cos n}{e^n} \geq 0$,

so that one can use the comparison test with $b_n = \frac{2}{e^n}$

Then $a_n \leq b_n$, whereas $\sum_{n=1}^{\infty} b_n$ converges (as a geometric series with the ratio $\frac{1}{e} < 1$)

$$\Longrightarrow \sum_{n=1}^{\infty} a_n \text{ converges}$$

③ $\sum_{n=1}^{\infty} \frac{\ln^2 n}{\sqrt{n}}$

Solution $\ln^2 n \geq 1$ for $n \geq 3 > e$,

therefore one can compare the original series with the entries $a_n = \frac{\ln^2 n}{\sqrt{n}}$ with $b_n = \frac{1}{\sqrt{n}}$

$a_n \geq b_n$ for $n \geq 3$,

whereas $\sum_{n=1}^{\infty} b_n = \infty$ (it is polynomial with $p = \frac{1}{2}$)

$\Rightarrow \sum_{n=1}^{\infty} a_n$ also diverges

4) $$\sum_{n=1}^{\infty} \frac{2n^2+n+1}{3n^4+n^2}$$

Solution Here we use the limit comparison test: the leading terms in the numerator and the denominator are $2n^2$ and $3n^4$, respectively, so that we take

$$b_n = \frac{n^2}{n^4} = \frac{1}{n^2}$$

Then
$$\lim_{n \to \infty} \frac{a_n}{b_n} = \lim_{n} \frac{(2n^2+n+1)/n^2}{(3n^4+n^2)/n^4}$$
$$= \lim \frac{\left(2+\frac{1}{n}+\frac{1}{n^2}\right)}{\left(3+\frac{1}{n^2}\right)} = \frac{2}{3}$$

Since $\sum b_n$ is convergent (as polynomial with $p=2$), $\sum a_n$ is also convergent

(5) $$\sum_{n=1}^{\infty} \frac{e^{n+1}+1}{ne^n+1}$$

Solution Here the leading terms e^{n+1} and ne^n, so that we take $b_n = \frac{e^{n+1}}{ne^n} = \frac{e}{n}$

Then $\lim_{n\to\infty} \frac{b_n}{a_n} = 1$

Since $\sum_{n=1}^{\infty} b_n = \sum_{n=1}^{\infty} \frac{e}{n}$ diverges, (as a constant multiple of the harmonic series),

$\sum_{n=1}^{\infty} a_n$ also diverges

(6) Estimate the error in replacing the sum of the series

$\sum_{n=1}^{\infty} \frac{1}{5+n^2}$ with the sum of its first 10 entries

Solution

$$R_{10} = a_{11} + a_{12} + \ldots$$

$$= \frac{1}{5+11^2} + \frac{1}{5+12^2} + \ldots$$

is estimated by above with

$$\frac{1}{11^2} + \frac{1}{12^2} + \ldots,$$

which is a series with decreasing positive entries, whence by the integral test

$$R_{10} \le \frac{1}{11^2} + \frac{1}{12^2} + \ldots \le \int_{10}^{\infty} \frac{dx}{x^2} = \frac{1}{10}$$

In problems 7–9 study the convergence of the given series

7) $-\frac{2}{5}+\frac{4}{6}-\frac{6}{7}+\frac{8}{8}-\frac{10}{9}+ \cdots$

Solution. First find a formula for a_n. Presuming the first entry is a_1, the numerators are $2n$, and the denominators are $n+4$. Since the signs are alternating,

$$a_n = (-1)^n \frac{2n}{n+4}, \text{ so that}$$

$$|a_n| = \frac{2n}{n+4} \longrightarrow 2 \neq 0$$

$\Longrightarrow$ the series is <u>not</u> convergent

(8) $\sum (-1)^{n+1} \frac{1}{\sqrt{n}}$

Solution. This is an alternating series, $|a_n| = \frac{1}{\sqrt{n}}$, so that $|a_n|$ is 1) monotone

2) $|a_n| \to 0$

$\Rightarrow$ the series is convergent

(9) $\sum_{n=1}^{\infty} (-1)^n n e^{-n/3}$

Solution Here $|a_n| = ne^{-n/3}$, so

that $|a_n| = ne^{-n/3} = \frac{n}{e^{n/3}} \to 0$

by L'Hôpital's rule

Still one has to check the

monotonicity of $|a_n|$.

Consider $f(x) = xe^{-x/3}$, then

$f'(x) = e^{-x/3} - \frac{x}{3}e^{-x/3} = e^{-x/3}\left(1 - \frac{x}{3}\right)$,

so that $f'(x) \leq 0$ for $x \geq 3$,

and $|a_n|$ is monotone decreasing

for $n \geq 3$

$\Rightarrow$ the series is convergent

(10) How many entries of the series $\sum_{n=1}^{\infty} \frac{(-1)^n}{n^3}$ should one take to estimate the sum with the error $< \varepsilon = 0.001$

Solution For an alternating series which satisfies the conditions of the convergence test ($|a_n| \to 0$ and $|a_n|$ monotone)

$$|R_n| \le |a_{n+1}|,$$ so that one has to solve the inequality

$$|a_{n+1}| \le \varepsilon$$

$$\frac{1}{(n+1)^3} \le 0.001$$

$$(n+1) \ge 10$$

The minimal n that satisfies this condition is $n = 9$