DGD 4

(1) If the n-th partial sum of a series $\sum_{n=1}^{\infty} a_n$ is $s_n = 3 - n2^{-n}$, find a_n and $\sum_{n=1}^{\infty} a_n$

Solution (a) $a_n = s_n - s_{n-1}$

$$= (3 - n\cdot 2^{-n}) - (3 - (n-1)2^{-n+1}) =$$

$$= (n-1)\cdot 2 \cdot 2^{-n} - n\cdot 2^{-n}$$

$$= (2n - 2 - n)\cdot 2^{-n} = (n-2)\, 2^{-n}$$

(b) $$\sum_{n=1}^{\infty} a_n = \lim_{n\to\infty} s_n =$$

$$= \lim_{n\to\infty} (3 - n\cdot 2^{-n})$$

$$= 3 - \lim_{n\to\infty} \frac{n}{2^n} = 3,$$

because $\lim_{n\to\infty} \frac{n}{2^n} = 0$

(by this time students should know that polynomial functions grow slower than exponential ones; otherwise use L'Hôpital)

(2) Is the following series geometric? Determine whether it is convergent or divergent. If it is convergent, find the sum:

$$\sum_{n=0}^{\infty} \frac{3^{n+1}}{(-2)^{3n}}$$

Solution: The n-th term is $a_n = \dfrac{3^{n+1}}{(-2)^{3n}}$,

whence $$\frac{a_{n+1}}{a_n} = \frac{3^{n+2}/(-2)^{3n+3}}{3^{n+1}/(-2)^{3n}} = -\frac{3}{8},$$

which does not depend on n. Thus, the series is geometric with the ratio $r = -\frac{3}{8}$. Since $|r| = \frac{3}{8} < 1$, the series is convergent, and

$$\sum_{n=0}^{\infty} a_n = \frac{a_0}{1-r} = \frac{3}{1+\frac{3}{8}} = \frac{3}{11/8}$$

$$= \frac{24}{11}$$

③ Express the number as a ratio of integers : $1.2\overline{36}$

<u>Solution</u> : $1.2\overline{36} = 1.2363636\ldots$

$$= 1.2 + \frac{36}{10^3} + \frac{36}{10^5} + \ldots$$

$$= 1.2 + \frac{36}{10^3} \frac{1}{1 - \frac{1}{100}}$$

$$= 1.2 + \frac{36}{10^3} \frac{10^2}{99} =$$

$$= 1.2 + \frac{36}{990}$$

$$= 1.2 + \frac{4}{110}$$

$$= 1 + \frac{2}{10} + \frac{4}{110} = \frac{110 + 22 + 4}{110}$$

$$= \frac{136}{110} = \frac{68}{55}$$

(4) Determine whether the series is convergent or divergent. If it is convergent, find its sum: $\frac{1}{3}+\frac{1}{6}+\frac{1}{9}+\frac{1}{12}+\dots$

Solution: $a_n = \frac{1}{3n}$, whence

$$\sum_{n=1}^{\infty} a_n = \sum_{n=1}^{\infty} \frac{1}{3n} =$$

$$= \frac{1}{3} \sum_{n=1}^{\infty} \frac{1}{n}$$

The series $\sum_{n=1}^{\infty} \frac{1}{n}$ is harmonic and diverges $\Longrightarrow$

$\sum_{n=1}^{\infty} a_n$ also diverges

5) Determine whether the series is convergent or divergent. If it is convergent, find its sum: $\sum_{n=1}^{\infty} \frac{2^n + 4^n}{e^n}$

Solution Here

$$a_n = \frac{2^n + 4^n}{e^n} = \frac{4^n\left(1+\left(\frac{2}{4}\right)^n\right)}{e^n}$$

$$= \left(\frac{4}{e}\right)^n \cdot \left[1 + \left(\frac{1}{2}\right)^n\right]$$

$\left(\frac{1}{2}\right)^n \to 0 \implies$

$1 + \left(\frac{1}{2}\right)^n \to 1$, whereas

$\left(\frac{4}{e}\right)^n \to \infty$, because $e \cong 2.72 < 4$

Therefore, $a_n \not\to 0$, hence the series is divergent

⑥ Use the integral test to determine whether the series is convergent or divergent:
$$\sum_{n=1}^{\infty} \frac{n}{n^2+4}$$

Solution. Consider the function $f(x) = \frac{x}{x^2+4}$. In order to apply the integral test the function has to be non-negative (which is the case), and monotone. To check the latter, find

$$f'(x) = \frac{x' \cdot (x^2+4) - x \cdot (x^2+4)'}{(x^2+4)^2} = \frac{x^2+4-2x^2}{(x^2+1)^2}$$

$$= \frac{4-x^2}{(x^2+1)^2},$$ which is ≤ 0 for $x \geq 2$, so that the integral criterion is applicable (remind that it is enough that f be monotone starting from a certain point)

Now, $\int_1^{\infty} \frac{x}{x^2+4}\, dx = \infty$

(either by comparison with $\int_1^{\infty} \frac{x}{5x^2}\, dx = \infty$ or by change of variables $t = x^2+4$), so that the series diverges

(7) Use the integral test to determine whether the series is convergent or divergent: $\sum_{n=2}^{\infty} \frac{1}{n \ln^2 n}$

Solution. Consider $f(x) = \frac{1}{x \ln^2 x}$

Then (a) $f(x) \geq 0$ for $x \geq 2$

(b) $f'(x) = \frac{-\ln^2 x - 2\ln x}{x \ln^2 x}$

so that $f'(x) \leq 0$ for $x \geq 2$

therefore the integral test is applicable.

Now,
$$\int_2^{\infty} f(x)\,dx = \int_2^{\infty} \frac{dx}{x \ln^2 x} =$$
$$= \int_2^{\infty} \frac{d(\ln x)}{\ln^2 x} = \int_{\ln 2}^{\infty} \frac{dt}{t^2} =$$
$$= \left(-\frac{1}{t}\right)\Bigg|_{t=\ln 2}^{t=\infty} = \frac{1}{\ln 2} < \infty$$

Therefore, the series is convergent.

(8) How many terms of the series

$$\sum_{n=1}^{\infty} a_n, \qquad a_n = \frac{1}{n^2}$$

does one need to take to find its sum within 0.01

Solution The function $f(x) = \frac{1}{x^2}$ is positive and monotone, so that by the integral test the series is convergent.

The remainder $R_n = a_{n+1} + a_{n+2} + \dots$ satisfies the inequality $R_n \le \int_n^{\infty} f(x)\,dx$ and we have to find the minimal n such that

$$\int_n^{\infty} f(x)\,dx \le 0.01$$

Since

$$\int_n^{\infty} f(x)\,dx = \int_n \frac{dx}{x^2} = \left(-\frac{1}{x}\right)\Big|_{x=n}^{\infty} = \frac{1}{n}$$

we arrive at the inequality

$$\frac{1}{n} \le 0.01$$

whence $n = 100$