OGD 3

① Find the first 5 entries of the sequence

$$a_n = \frac{(-1)^n n}{n! + 1}$$

Solution: Remind the definition of the factorial:

$$n! = 1 \cdot 2 \cdot \ldots \cdot n \; ;$$

$$1! = 1, \quad 2! = 2, \quad 3! = 2 \cdot 3 = 6,$$

$$4! = 3! \cdot 4 = 24, \quad 5! = 4! \cdot 5 = 120$$

Therefore,

$$a_1 = \frac{-1}{1+1} = -\frac{1}{2}$$

$$a_2 = \frac{(-1)^2 \cdot 2}{2! + 1} = \frac{2}{3}$$

$$a_3 = -\frac{3}{7}$$

$$a_4 = \frac{4}{25}, \quad a_5 = -\frac{5}{121}$$

(2) Write the first 5 entries of the sequence $a_1 = 6$, $a_{n+1} = \frac{a_n}{n}$ and find a general formula for a_n.

Solution:

$$a_2 = \frac{a_1}{1} = 6$$

$$a_3 = \frac{a_2}{2} = \frac{6}{2} = 3$$

$$a_4 = \frac{a_3}{3} = \frac{3}{3} = 1$$

$$a_5 = \frac{a_4}{4} = \frac{1}{4}$$

general formula:

$$a_n = \frac{1}{n-1} a_{n-1} = \frac{1}{(n-1)(n-2)} a_{n-2} = \dots$$

$$= \frac{1}{(n-1)(n-2)\cdots 2} a_2$$

$$= \frac{1}{(n-1)(n-2)\cdots 1} \cdot a_1 = \frac{6}{(n-1)!}$$

Remark, $0! = 1$

(3) Find a general formula for the sequence

$$\frac{1}{2}, -\frac{4}{3}, \frac{9}{4}, -\frac{16}{5}, \frac{25}{6}, \dots$$

Solution. Consider separately the numerator and the denominator.

Numerator: $1, 4, 9, 16, 25$ – recognize the sequence of squares n^2

Denominator: $2, 3, 4, 5, 6$ – recognize the arithmetic progression $n+1$

The sign is alternating $\Rightarrow$ add the multiplier either $(-1)^n$ if a_1 is negative, or $(-1)^{n+1}$ if $a_1 > 0$

Finally, $a_n = (-1)^{n+1} \frac{n^2}{n+1}$

④ Find whether the sequence converges or diverges. If it converges, find the limit:

$$a_n = \frac{3 + 5n^2}{n + n^2}$$

Solution. Separate the leading terms in the numerator: $5n^2$

and in the denominator: n^2

$$a_n = \frac{5n^2\left(1 + \frac{3}{5n^2}\right)}{n^2\left(1 + \frac{n}{n^2}\right)} =$$

$$= 5 \cdot \frac{1 + 3/(5n^2)}{1 + 1/n}$$

The use $\frac{1}{n} \to 0 \Rightarrow \frac{1}{n^2} \to 0$

Therefore

$$\frac{1 + 3/(5n^2)}{1 + 1/n} \to \frac{1 + 0}{1 + 0} = \frac{1}{1}$$

Finally, $\lim a_n = 5 \cdot \lim \frac{1 + 3/(5n^2)}{1 + 1/n} = 5$

(5) $a_n = \frac{4^n}{1+9^n}$

Solution. Use the squeeze theorem:

$$0 \le a_n \le \frac{4^n}{9^n} = \left(\frac{4}{9}\right)^n$$

The function $f(x) = \left(\frac{4}{9}\right)^x$ has the property $f(x) \xrightarrow[x \to \infty]{} 0$

because $0 < \frac{4}{9} < 1 \Rightarrow$

$$\lim_{n\to\infty} \left(\frac{4}{9}\right)^n = 0 \Rightarrow$$

$$\lim_{n\to\infty} a_n = 0$$

⑥ $a_n = e^{-\frac{1}{\sqrt{n}}}$

Solution. $f(x) = \frac{1}{\sqrt{x}} \xrightarrow[x\to\infty]{} 0 \Rightarrow$

$\frac{1}{\sqrt{n}} \to 0 \Rightarrow -\frac{1}{\sqrt{n}} \to 0$

The function $g(x) = e^x$ is continuous $\Rightarrow$

$$\lim a_n = \lim e^{-\frac{1}{\sqrt{n}}} = e^{\lim\left(-\frac{1}{\sqrt{n}}\right)}$$

$$= e^0 = 1$$

7) $a_n = (-1)^{n+1} \frac{n}{n+\sqrt{n}}$

Solution. Use the leading terms (cf. 4): $n + \sqrt{n} = n\left(1 + \frac{\sqrt{n}}{n}\right)$

$= n\left(1 + \frac{1}{\sqrt{n}}\right)$,

where $\frac{1}{\sqrt{n}} \to 0$, so that

$1 + \frac{1}{\sqrt{n}} \to 1$

Therefore $|a_n| = \frac{n}{n+\sqrt{n}} = \frac{1}{1+\frac{1}{\sqrt{n}}} \to 1$

However, the sign of a_n alternates

$\Rightarrow$ the subsequence of even terms

$a_{2n} = -|a_{2n}| = -\frac{1}{1+\frac{1}{\sqrt{2n}}} \to -1$,

and the subsequence of odd terms

$a_{2n+1} = |a_{2n+1}| = \frac{1}{1+\frac{1}{\sqrt{2n+1}}} \to 1$

Therefore a_n does <u>not</u> converge

(8) $a_n = \ln(n+1) - \ln n$

Solution

$$a_n = \ln(n+1) - \ln n$$

$$= \ln \frac{n+1}{n} = \ln\left(1 + \frac{1}{n}\right)$$

$1 + \frac{1}{n} \to 1$, and

$x \mapsto \ln x$ is continuous $\Longrightarrow$

$$a_n = \ln\left(1 + \frac{1}{n}\right) \to \ln\left(\lim_n \left(1 + \frac{1}{n}\right)\right)$$

$$= \ln 1 = 0$$

(9) $a_n = \frac{\cos n}{2^n}$

Solution $-1 \le \cos n \le 1 \Rightarrow$

$$-\frac{1}{2^n} \le a_n \le \frac{1}{2^n}$$

Since $\frac{1}{2^n} \to 0$ (cf. 5),

also $-\frac{1}{2^n} \to 0 \Rightarrow$

$\lim a_n = 0$ by the squeeze theorem

(10) $a_n = \left(1 + \frac{2}{n}\right)^{3n}$

Solution Use that

$$\lim_{x \to \infty} \left(1 + \frac{1}{x}\right)^x = e$$

Then $x = \frac{n}{2} \to \infty$, and

$$a_n = \left(1 + \frac{1}{x}\right)^{6x} = \left(\left(1 + \frac{1}{x}\right)^x\right)^6 \to e^6$$

(11) $a_n = \dfrac{(\ln n)^2}{n}$

Solution Both the numerator and the denominator $\to \infty$

$\Rightarrow$ use the l'Hopital rule

$$\left((\ln x)^2\right)' = 2\frac{\ln x}{x}$$

$$x' = 1 \quad \Rightarrow$$

$$\lim_{x\to\infty} \frac{(\ln x)^2}{x} = \lim_{x\to\infty} \frac{2\frac{\ln x}{x}}{1}$$

$$= \lim_{x\to\infty} 2\frac{\ln x}{x}$$ (use l'Hopital agai.)

$$= \lim_{x\to\infty} \frac{2}{x} = 0$$

$$\Rightarrow \lim \frac{(\ln n)^2}{n} = 0$$

(12) $a_n = \sqrt[n]{n}$

Solution Use

$$\ln a_n = \ln (n)^{1/n} = \frac{1}{n} \ln n$$

$$= \frac{\ln n}{n}$$

Since $\frac{\ln n}{n} \to 0$ (see 11)

$$a_n = e^{\frac{\ln n}{n}} \to e^0 = 1$$

because the function $x \mapsto e^x$ is continuous