OGD 2

① Find the average value of the function $f(x) = \frac{x}{\sqrt{3+x^2}}$ on the interval $[-1, 3]$

Solution: First find the integral

$$I = \int_{-1}^{3} \frac{x}{\sqrt{3+x^2}}\, dx = \frac{1}{2} \int_{-1}^{3} \frac{d(3+x^2)}{\sqrt{3+x^2}}$$

$$= \frac{1}{2} \int_{-1}^{0} \frac{d(3+x^2)}{\sqrt{3+x^2}} + \frac{1}{2} \int_{0}^{3} \frac{d(3+x^2)}{\sqrt{3+x^2}}$$

split the integration interval into the monotonocity intervals of $3+x^2$

$$= -\frac{1}{2} \int_{3}^{4} \frac{dt}{\sqrt{t}} + \frac{1}{2} \int_{3}^{12} \frac{dt}{\sqrt{t}} = \frac{1}{2} \int_{4}^{12} \frac{dt}{\sqrt{t}}$$

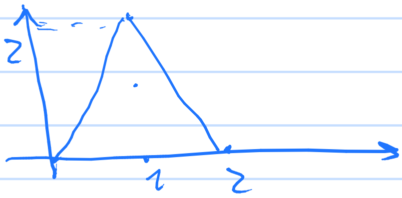
$$= t^{1/2} \Big|_{t=4}^{t=12} = \sqrt{12} - \sqrt{4} = 2(\sqrt{3} - 1)$$

Therefore,

$$\overline{f} = \frac{I}{\text{length}\,[-1, 3]} = \frac{I}{4} = \frac{\sqrt{3}-1}{2}$$

2) Find the centroid of the triangle with the vertices $(0,0)$, $(1,2)$, $(2,0)$

<u>Solution</u>



The triangle S has base 2 and height 2, whence its area is

$$A = \frac{1}{2}\cdot 2 \cdot 2 = 2$$

The triangle is enclosed between the graphs of the functions

$$f(x) = \begin{cases} 2x, & 0 \le x \le 1 \\ 4-2x, & 1 \le x \le 2 \end{cases} \quad \text{and} \quad g(x) = 0$$

Therefore, $\bar{x} = \frac{I_x}{A}$, $\bar{y} = \frac{I_y}{A}$, where

$$I_x = \int_0^2 x\,(f(x)-g(x))\,dx = \int_0^1 x\cdot 2x\,dx + \int_1^2 x\,(4-2x)\,dx$$

$$= \frac{2}{3}x^3\Big|_{x=0}^{x=1} + \left(2x^2 - \frac{2}{3}x^3\right)\Big|_{x=1}^{x=2}$$

$$= \frac{2}{3} + 8 - \frac{16}{3} - 2 + \frac{2}{3} = 2 \Rightarrow \bar{x} = \frac{I_x}{A} = \frac{2}{2} = 1$$

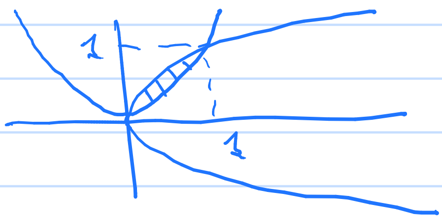
$$I_y = \frac{1}{2}\int_0^2 \left(f^2(x) - g^2(x)\right) dx = \frac{1}{2}\int_0^1 4x^2\,dx + \frac{1}{2}\int_1^2 (4-2x)^2\,dx$$

$$= \frac{2}{3}x^3\Big|_{x=0}^{x=1} + 2\int_1^2 (x-2)^2\,dx = \frac{2}{3} + \frac{2}{3}(x-2)^3\Big|_{x=1}^{x=2} = \frac{2}{3} + \frac{2}{3} = \frac{4}{3}$$

$$\Rightarrow \bar{y} = \frac{I_y}{A} = \frac{4/3}{2} = \frac{2}{3}$$

(3) Find the centroid of the domain bounded by $y = x^2$ and $x = y^2$

Solution



The domain can be described as $\{(x,y) : 0 \le x \le 1,\ \underbrace{x^2}_{g(x)} \le y \le \underbrace{\sqrt{x}}_{f(x)}\}$

Its area is $A = \int_0^1 (f(x) - g(x))\,dx = \int_0^1 (\sqrt{x} - x^2)\,dx =$

$$= \left(\frac{2}{3}x^{3/2} - \frac{1}{3}x^3\right)\Big|_{x=0}^{x=1} = \frac{2}{3} - \frac{1}{3} = \frac{1}{3}$$

$$I_x = \int_0^1 x\,(f(x) - g(x))\,dx = \int_0^1 x(\sqrt{x} - x^2)\,dx$$

$$= \left(\frac{2}{5}x^{5/2} - \frac{1}{4}x^4\right)\Big|_{x=0}^{x=1} = \frac{2}{5} - \frac{1}{4} = \frac{3}{20}$$

$$I_y = \frac{1}{2}\int_0^1 (f^2(x) - g^2(x))\,dx = \frac{1}{2}\int_0^1 (x - x^4)\,dx$$

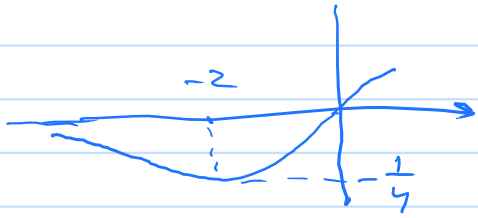
$$= \frac{1}{2}\left(\frac{x^2}{2} - \frac{x^5}{5}\right)\Big|_{x=0}^{x=1} = \frac{1}{2}\left(\frac{1}{2} - \frac{1}{5}\right) = \frac{3}{20}$$

Thus, $\overline{x} = \frac{I_x}{A} = \frac{3/20}{1/3} = \frac{9}{20}$

$$\overline{y} = \frac{I_y}{A} = \frac{3/20}{1/3} = \frac{9}{20}$$

(4) Find whether the integral is convergent or divergent: $I = \int_{-\infty}^{0} \frac{z}{z^2+4}\,dz$

Solution



$$\int_{-\infty}^{0} \frac{z}{z^2+4}\,dz = -\frac{1}{2}\int_{0}^{\infty} \frac{d(z^2)}{z^2+4} = -\frac{1}{2}\int_{0}^{\infty} \frac{dt}{t+4}$$

$$= -\frac{1}{2} \ln(t+4)\Big|_{t=0}^{t=\infty}$$

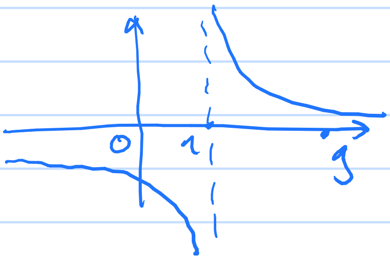
Since $\ln t \xrightarrow[t\to\infty]{} \infty$, the integral diverges

(5) Is the following integral divergent or convergent?

$$I = \int_0^9 \frac{dx}{\sqrt[3]{x-1}};$$

Find its value if it is convergent.

<u>Solution.</u> The function $f(x) = \frac{1}{\sqrt[3]{x-1}}$ is not defined at $x = 1$, so that the



integration interval has to be split into subintervals $[0,1]$, $[1,9]$

$$I = \int_0^1 \frac{dx}{\sqrt[3]{x-1}} + \int_1^9 \frac{dx}{\sqrt[3]{x-1}}$$

$$= \frac{3}{2}(x-1)^{2/3}\Big|_{x=0}^{x=1} + \frac{3}{2}(x-1)^{2/3}\Big|_{x=1}^{9}$$

$(x-1)^{2/3} = 0$ for $x = 1$, whence the integral is convergent, and its value is

$$I = \left(0 - \frac{3}{2}\right) + \left(\frac{3}{2}\cdot 4 - 0\right) = \frac{9}{2}$$

⑥ By using the comparison test find whether the integral $I = \int_1^\infty \frac{x+1}{\sqrt{x^4-x}}\,dx$

is divergent or convergent

<u>Solution</u> This integral is improper for two reasons: the integration interval is infinite and there is a discontinuity at $x=1$

Therefore, one should split the integration interval into two by taking an intermediate point, for instance, $[1,2]$ and $[2,\infty)$ and investigate each interval separately

$$x^4 - x = x(x^3-1) = x(x-1)(x^2+x+1)$$

$$I_1 = \int_1^2 \frac{(x+1)\,dx}{(x^4-x)^{1/2}} = \int_1^2 \underbrace{\frac{x+1}{(x(x^2+x+1))^{1/2}}}_{\text{bdd. on } [1,2]} \cdot \frac{1}{\sqrt{x-1}}\,dx$$

$$\int_1^2 \frac{dx}{\sqrt{x-1}} = 2\sqrt{x-1}\,\Big|_{x=1}^{x=2} \quad \text{is convergent}$$

$$\Longrightarrow I_1 \text{ is convergent}$$

$$I_2 = \int_2^\infty \frac{x+1}{(x^4-x)^{1/2}}\,dx = \int_2^\infty \underbrace{\frac{x(x+1)}{x^2\cdot\left(1-\frac{1}{x^3}\right)^{1/2}}}_{\xrightarrow[x\to\infty]{} 1} \frac{1}{x}\,dx,$$

$$\int_2^\infty \frac{dx}{x} = \ln x\,\Big|_{x=2}^{\infty} \quad \text{is divergent}$$

$\Rightarrow I_2$ diverges $\Longrightarrow I = I_1 + I_2$ diverges