Actuarial Mathematics I
Actu 257

José Garrido
Department of Mathematics and Statistics
Concordia University

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Chapter 0

Introduction

0.1 Actuarial Present Value

Theory of Interest suggests methods to evaluate the *present value* of known payments P at fixed times, e.g.,

$$\begin{array}{ll} P\,v^n & \text{for a single payment in } n \text{ years,} \\ P\,\bar{a}_{\overline{n}|} & \text{for } n \text{ continuous payments at rate } P \text{ per year.} \end{array}$$

In Insurance or Pension applications, n is *unknown*, for example:

$$\left.\begin{array}{l} P\,v^T \\ P\,\bar{a}_{\overline{T}|} \end{array}\right\} \quad \text{where } T \text{ is the future lifetime of a person } \Rightarrow \text{ a random variable .}$$

This leads to the concept of *actuarial present value*:

$$\mathbb{E}[P\,v^T] = P\,\mathbb{E}[v^T] = P\,\mathbb{E}[e^{-\delta\,T}] = P\,M_T(-\delta)\ ,$$

or

$$\mathbb{E}[\bar{a}_{\overline{T}|}] = \mathbb{E}\left[\frac{1-v^T}{\delta}\right] = \frac{1-M_T(-\delta)}{\delta}\ .$$

0.2 Contingencies

Actuarial present values are function of the distribution of certain random variables, as the payments are *contingent* on the value of these random variables. For instance:

Life Contingencies - distribution of lifetime T,
Financial Contingencies - distribution of interest rates i,
Economical Contingencies - distribution of salary, inflation, etc.

Chapter 1

Survival Distributions and Life Tables

1.1 Introduction

Observe a newborn until death, at age X, say. This age at death is a random variable characterized by

$$F(x) = \mathbb{P}\{X \leq x\} , \qquad x \geq 0 .$$

Problems of Interest:

Modelling: For a given F calculate certain actuarial functions (present value, annuities, accumulated values, ...) $\longrightarrow$ Life Contingencies (Course 3).

Estimation: For observed ages at death $X_1, \ldots, X_n$ obtain a good estimate of F $\longrightarrow$ Survival Analysis (Course 4).

Section 3.3 of the textbook, "Life Tables", gives a brief introduction to the subject of estimation.

1.2 Death Probabilities

- Survival of a newborn to age x:

$$\mathbb{P}\{X > x\} = 1 - F(x) = s(x) , \qquad x \geq 0 .$$

- Death of a newborn before age y, given that the newborn survived to age $0 \leq x < y$:

$$\begin{aligned}\mathbb{P}\{X \leq y \,|\, X > x\} &= \frac{\mathbb{P}\{x < X \leq y\}}{\mathbb{P}\{X > x\}} \\ &= \frac{[F(y) - F(x)]}{s(x)} = \frac{[s(x) - s(y)]}{s(x)}\,, \qquad x \geq 0\,.\end{aligned}$$

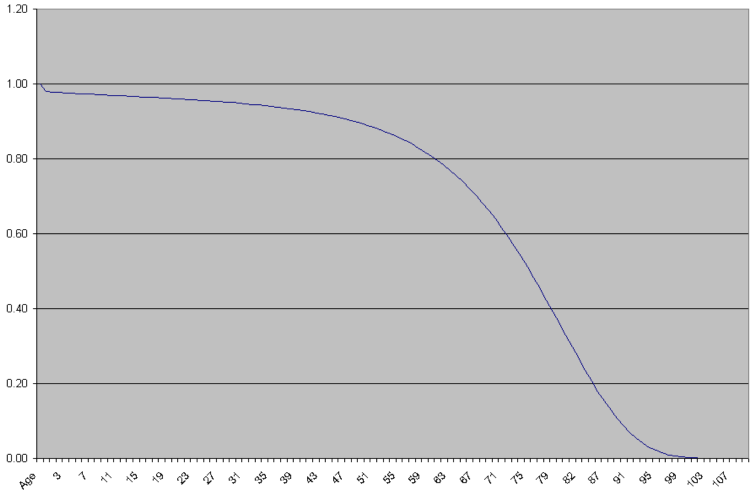

Figure 1.1: Survival Function of a Newborn

The quantity of interest is $X - x$, not X, as newborns do not buy insurance. Given that $X > x$, denote by $T(x) = X - x$ the future lifetime of a policyholder now age x (with $T(0) = X$). Its cumulative distribution function (cdf) is given by:

$$\begin{aligned}\mathbb{P}\{T(x) \leq t\} &= \mathbb{P}\{X - x \leq t \,|\, X > x\}\,, \qquad t \geq 0 \\ &= \frac{[s(x) - s(x+t)]}{s(x)} \\ &:= {}_tq_x\,.\end{aligned}$$

It is the probability that a newborn who survived to age x will not survive to age $x + t$, i.e., a life age x [henceforth denoted by (x)] dies before the end of t years.

The corresponding probability of survival is given by:

$$ {}_tp_x := 1 - {}_tq_x \ , \qquad t \geq 0 \ . $$

Remark 1.1

$$ {}_tp_0 = s(t) \qquad \text{and} \qquad {}_tp_x = \frac{s(x+t)}{s(x)} = \frac{{}_{x+t}p_0}{{}_xp_0} \ . $$

- The probability that (x) dies between ages $x+t$ and $x+t+u$:

$$ \begin{aligned} \mathbb{P}\{t < T(x) \leq t+u\} &= {}_{t+u}q_x - {}_tq_x \\ &= {}_tp_x - {}_{t+u}p_x \\ &:= {}_{t|u}q_x \ . \end{aligned} \tag{1.1} $$

  (1.1) can be interpreted as a probability of survival to age $x+t$ but not to age $x+t+u$.

  It can also be written as the probability

$$ {}_{t|u}q_x = {}_tp_x \, {}_uq_{x+t} \ , $$

  of survival to age $x+t$ followed by a death before age $x+t+u$.

Remark 1.2 The following notational conventions apply

$$ {}_1q_x := q_x \quad \text{and} \quad {}_{t|1}q_x := {}_{t|}q_x \ . $$

- All these probabilities can thus be written in terms of survival probabilities ${}_tp_x$, death probabilities ${}_tq_x$ or of the survival function. For instance:

$$ \begin{aligned} {}_{t|u}q_x &= {}_tp_x - {}_{t+u}p_x \\ &= \frac{s(x+t)}{s(x)} - \frac{s(x+t+u)}{s(x)} \\ &= \frac{[s(x+t) - s(x+t+u)]}{s(x)} \ . \end{aligned} $$

- q_x is the death probability of (x) for the next year, while ${}_tq_x$ is for the next t years (where usually $\frac{{}_tq_x}{t} > q_x$). Hence
$$\frac{{}_tq_x}{t}$$
is the annual average, that is the average death rate *per year* over the interval $[x, x+t]$. The corresponding *instantaneous rate* of mortality at age x is thus
$$\lim_{t\to 0}\frac{{}_tq_x}{t} = \lim_{t\to 0}\frac{[s(x)-s(x+t)]}{t\,s(x)} = \frac{-s'(x)}{s(x)} := \mu(x)\ ,$$
the *force of mortality.*

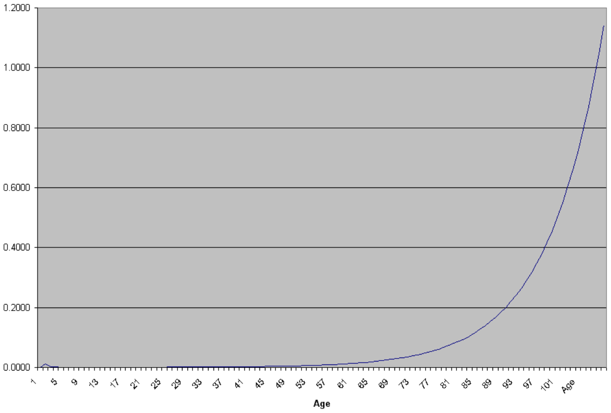

Figure 1.2: Force of Mortality

Since
$$\mu(x) = \frac{-s'(x)}{s(x)} = -\frac{d}{dx}\ln s(x)\ ,$$
then
$$-\int_x^{x+t}\mu(y)\,dy = \ln s(x+t) - \ln s(x)$$
and hence
$$\exp\{-\int_x^{x+t}\mu(y)\,dy\} = {}_tp_x\ ,$$
which means that all death probabilities can be expressed in terms of $\mu(x)$.

Example 1.1

$$\begin{aligned}
{}_tq_x &= 1 - {}_tp_x = 1 - \exp\{-\int_x^{x+t} \mu(y)\,dy\}\ , \\
{}_{t|u}q_x &= {}_tp_x\ {}_uq_{x+t} \\
&= \exp\{-\int_x^{x+t} \mu(y)\,dy\} \left[1 - \exp\{-\int_{x+t}^{x+t+u} \mu(y)\,dy\}\right] \\
&= {}_tp_x - {}_{t+u}p_x\ .
\end{aligned}$$

Also $\mu(x) = \frac{-s'(x)}{s(x)}$ implies

$$\begin{aligned}
-s'(x) &= s(x)\,\mu(x) \\
\Rightarrow \quad s(x) - s(x+t) &= \int_x^{x+t} s(y)\,\mu(y)\,dy \\
\Rightarrow \quad \frac{s(x) - s(x+t)}{s(x)} &= \int_x^{x+t} \frac{s(y)}{s(x)}\,\mu(y)\,dy \\
\Rightarrow \quad {}_tq_x &= \int_x^{x+t} {}_{y-x}p_x\,\mu(y)\,dy \\
&= \int_0^t {}_sp_x\,\mu(x+s)\,ds
\end{aligned}$$

Exercise 1.1 Show that

$$ {}_{t|u}q_x = \int_t^{t+u} {}_sp_x\,\mu(x+s)\,ds\ . $$

- If ${}_tq_x$ is the cumulative distribution function (cdf) of $T(x)$ and if it admits a density function, then for a fixed $x \geq 0$ it is given by:

$$\begin{aligned}
\frac{\partial}{\partial t}{}_tq_x &= \frac{\partial}{\partial t}\left[1 - \exp\{-\int_x^{x+t} \mu(y)\,dy\}\right] \\
&= -\exp\{-\int_x^{x+t} \mu(y)\,dy\}\left[-\mu(x+t)\right] \\
&= {}_tp_x\,\mu(x+t)\ , \qquad t > 0\ .
\end{aligned}$$

  The Illustrative Life Table in Appendix 2A gives a graph of ${}_tp_0\,\mu(t) = s(t)\,\mu(t)$:

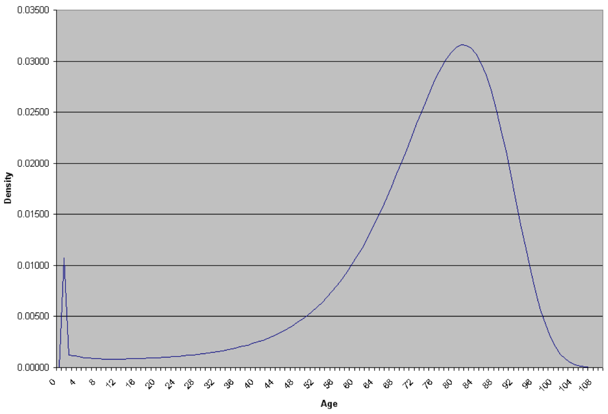

Figure 1.3: Density Function of the Future Lifetime of a Newborn

1.3 Life Tables

Statistical estimation of $s(x)$: observe l_0 newborns and denote by

$$\begin{aligned}\mathcal{L}(x) &= \#\text{ who survive to age } x \quad \Rightarrow \quad \text{a random variable ,}\\ &= \sum_{j=1}^{l_0} I_j \ , \quad \text{say ,}\end{aligned}$$

where

$$I_j = \begin{cases} 1 & \text{if the } j\text{-th individual is alive at age} x \ , \\ 0 & \text{otherwise .}\end{cases}$$

for $j = 1, \ldots, l_0$.

Now

$$\mathbb{E}[I_j] = 1\, s(x) + 0\,[1 - s(x)] = s(x) = {}_xp_0 \ ,$$

implies that

$$\mathbb{E}[\mathcal{L}(x)] = \sum_{j=1}^{l_0} s(x) = l_0\, s(x) = l_0\, {}_xp_0 := l_x \ ,$$

is the expected number alive at age x. Note that with this definition

$${}_xp_0 = \frac{l_x}{l_0} \ .$$

Remark 1.3 $\mathcal{L}(x) \sim Bi\big(l_0, s(x)\big)$ if all the I_j's are independent. Then

$$\mathbb{V}[\mathcal{L}(x)] = l_0\, s(x)\, [1 - s(x)] = l_0\, {}_x p_0\, {}_x q_0 \ .$$

Example 1.2 (US Life Table 1979-81): Table 3.2 of the textbook.

Assume that $l_0 = 100,000$ newborns are observed. The number of survivors at age 1, $\mathcal{L}(1)$ is a random variable with expected value given by:

$$\begin{aligned} \mathbb{E}[\mathcal{L}(1)] &= 100,000\, s(1) \\ &= 100,000\, {}_1 p_0 \\ &= 100,000\, \frac{l_1}{l_0} \\ &= l_1 = 98,740 \end{aligned}$$

Note that with the 1959-61 US Life Table (see Jordan, p.11) the corresponding value is

$$l_1 = 97,408 \ .$$

All the functions defined above can now be obtained in terms of of a table of l_x values, for integer ages x:

$$l_x = l_0\, s(x) \quad \Rightarrow \quad s(x) = \frac{l_x}{l_0} \ .$$

For example, for integer ages x:

$$\begin{aligned} {}_n p_x &= \frac{s(x+n)}{s(x)} = \frac{l_{x+n}}{l_x} \ , \\ {}_{n|m} q_x &= \frac{s(x+n) - s(x+n+m)}{s(x)} = \frac{l_{x+n} - l_{x+n+m}}{l_x} \ , \end{aligned}$$

for integers n and m.

Also

$$\mu(x) = -\frac{d}{dx} \ln s(x) = -\frac{d}{dx} \ln l_x \ ,$$

which is found numerically, as l_x is only available for integers x (see Section 3.7).

Now let $\mathcal{D}(x)$ denote the number of deaths between ages x and $x+1$ among the original l_0 newborns. Then

$$\begin{aligned} \mathcal{L}(x+1) &= \mathcal{L}(x) - \mathcal{D}(x) \\ \Rightarrow \quad l_{x+1} &= l_x - \mathbb{E}[\mathcal{D}(x)] \\ \Rightarrow \quad \mathbb{E}[\mathcal{D}(x)] &= l_x - l_{x+1} := d_x \ . \end{aligned}$$

Similarly ${}_nd_x := l_x - l_{x+n}$ (with ${}_1d_x = d_x$).

Exercise 1.2 Since $\mathcal{L}(x) \sim Bi\big(l_0, s(x)\big)$ prove that :

1. $\mathcal{D}(x) \sim Bi(l_0, {}_{x|1}q_0)$,
2. given that $\mathcal{L}(x) = l_x$, the conditional distribution of $\mathcal{D}(x) \sim Bi(l_x, q_x)$.

Example 1.3 (Financial Application): How much does it costs now to promise $1,000$ people age 35 a \$1-payment if alive at age 55? [You can use $i = 0.03$, $l_{35} = 9,373,807$ and $l_{55} = 8,331,317$].

Solution: Without mortality the solution is simply

$$1000(1.03)^{-20} = \$553.68 \ ,$$

or \$0.55 each.

With mortality, the \$1-payment is only made with probability

$$\frac{l_{55}}{l_{35}} = \frac{8,331,317}{9,373,807} = 0.88879 \ ,$$

(which means a \$0-payment with probability 0.11120).

On average the total payment in 20 years is thus \$888.79, which has a present value of

$$888.79(1.03)^{-20} = \$492.10 \ ,$$

or \$0.49 each.

You can check that $9,373,807 \times 0.4921 = \$4,612,850$, which after 20 years will accumulate to

$$4,612,850.425(1.03)^{20} = \$8,331,319.894 \ ,$$

or \$1 per survivor.

1.4 The Complete Expectation of Life

Call G the cdf of $T(x)$:

$$G(x) = \mathbb{P}\{T(x) \le t\} = {}_tq_x = \int_0^t {}_sp_x\, \mu(x+s)\, ds\ , \qquad 0 \le t \le \omega - x\ ,$$

which implies a density of

$$g(t) = G'(t) = \frac{\partial}{\partial t} {}_tq_x = {}_tp_x\, \mu(x+t)\ , \qquad 0 \le t \le \omega - x\ .$$

Hence the expectation of the future lifetime $T(x)$ is given by

$$\begin{aligned}
\mathring{e}_x = \mathbb{E}[T(x)] &= \int_0^{\omega-x} t\, g(t)\, dt \\
&= \int_0^{\omega-x} t\, {}_tp_x\, \mu(x+t)\, dt \\
&= \int_0^{\omega-x} t\, \frac{\partial}{\partial t}(1 - {}_tp_x)\, dt \\
&= -t\, {}_tp_x\big|_0^{\omega-x} + \int_0^{\omega-x} {}_tp_x\, dt\ , \qquad \text{by parts}\ , \\
&= \int_0^{\omega-x} {}_tp_x\, dt\ .
\end{aligned}$$

The expected future lifetime $\mathring{e}_x$ is a useful index to compare the mortality experience of different populations. For instance:

US Life Table for White Males 1959-61	$\mathring{e}_0 = 67.55$,
US Life Table for Total Population 1979-81	$\mathring{e}_0 = 73.88$,
Canadian Population data - 1991 (Males)	$\mathring{e}_0 = 74.6$,
Canadian Population data - 1991 (Females)	$\mathring{e}_0 = 80.96$.

The variance of the future lifetime, around this $\mathring{e}_x$, can be found in a similar way:

$$\begin{aligned}
\mathbb{E}[T^2(x)] &= \int_0^{\omega-x} t^2\, {}_tp_x\, \mu(x+t)\, dt = \int_0^{\omega-x} t^2\, \frac{\partial}{\partial t}(-{}_tp_x)\, dt \\
&= -t^2\, {}_tp_x\big|_0^{\omega-x} + \int_0^{\omega-x} 2t\, {}_tp_x\, dt\ , \qquad \text{by parts}\ , \\
&= 2\int_0^{\omega-x} t\, {}_tp_x\, dt\ .
\end{aligned}$$

Hence

$$\mathbb{V}[T(x)] = 2\int_0^{\omega-x} t\,{}_tp_x\,dt - \mathring{e}_x^2\,.$$

Example 1.4 $T(x) \sim \exp(80-x)$, that is

$${}_tp_x\,\mu(x+t) = \frac{1}{(80-x)}e^{\frac{-t}{(80-x)}}\,, \qquad t > 0\,.$$

Verify that then:

$$\mathbb{E}[T(x)] = 80 - x \qquad \text{and} \qquad \mathbb{V}[T(x)] = (80-x)^2\,.$$

Commutation Functions: Define

$$T_x := \int_0^{\omega-x} t\,l_{x+t}\,\mu(x+t)\,dt\,.$$

This is the total *number of years* lived after age x by all the l_x lives (each dying at some posterior age $x+t$).

Then

$$\begin{aligned}\mathring{e}_x &= \int_0^{\omega-x} t\,{}_tp_x\,\mu(x+t)\,dt \\ &= \frac{1}{l_x}\int_0^{\omega-x} t\,l_{x+t}\,\mu(x+t)\,dt = \frac{T_x}{l_x}\,.\end{aligned}$$

Exercise 1.3 Prove that $T_x = \int_0^{\omega-x} l_{x+t}\,dt$ (by parts).

Similarly

$$\begin{aligned}\mathbb{V}[T(x)] &= 2\int_0^{\omega-x} t\,{}_tp_x\,dt - \mathring{e}_x^2 = \frac{2}{l_x}\int_0^{\omega-x}\int_0^t l_{x+t}\,ds\,dt \\ &= \frac{2}{l_x}\int_0^{\omega-x}\underbrace{\int_s^{\omega-x} l_{x+t}\,dt}_{T_{x+s}}\,ds - \mathring{e}_x^2 \\ &= \frac{2}{l_x}\int_0^{\omega-x} T_{x+s}\,ds - \mathring{e}_x^2\,.\end{aligned}$$

Consider now the survival over a finite age interval $[x, x+n)$, for $n \le w-x$, then:

$$\mathbb{E}\left[\min\bigl(n, T(x)\bigr)\right] = \int_0^n t\,{}_tp_x\,\mu(x+t)\,dt + n\,{}_np_x := \mathring{e}_{x:\overline{n}|}\,.$$

Exercise 1.4 Prove that $\mathring{e}_{x:\overline{n}|} = \frac{T_x - T_{x+n}}{l_x} = \int_0^n {}_tp_x\,dt$.

1.5 The Curtate Expectation of Life

Often, in measuring lifetime, only completed years are recorded. Then the corresponding expected lifetime is called the *curtate expectation of life* (not necessarily an integer) and denoted by

$$e_x = \mathbb{E}[K(x)]\,, \qquad \text{where } K(x) = \lfloor T(x) \rfloor\,.$$

For example, (x) who survives to $x+k$ but dies before completing the $(x+k+1)$st anniversary will only contribute k years to e_x, with probability ${}_kp_x\, q_{x+k} = {}_kp_x - {}_{k+1}p_x = {}_{k|}q_x$.

$$\begin{aligned} e_x &= \mathbb{E}[K(x)] = \sum_{k=0}^{\omega-x-1} k\, {}_kp_x\, q_{x+k} \\ &= \sum_{k=1}^{\omega-x-1} k\left({}_kp_x - {}_{k+1}p_x\right) \\ &= {}_1p_x - {}_2p_x + 2\,{}_2p_x - 2\,{}_3p_x + 3\,{}_3p_x - \ldots \\ &= [{}_1p_x + 2\,{}_2p_x + 3\,{}_3p_x + \ldots + (\omega-x-1)\,{}_{\omega-x-1}p_x] \\ &\qquad -[{}_2p_x + 2\,{}_3p_x + \ldots + (\omega-x-1)\,{}_{\omega-x}p_x] \\ &= \sum_{k=1}^{\omega-x-1} {}_kp_x\,. \end{aligned}$$

Also

$$\begin{aligned} \mathbb{E}\left\{[K(x)]^2\right\} &= \sum_{k=1}^{\omega-x-1} k^2\, {}_kp_x\, q_{x+k} \\ &= \sum_{k=1}^{\omega-x-1} k^2\left({}_kp_x - {}_{k+1}p_x\right) \\ &= [p_x + 4\,{}_2p_x + 9\,{}_3p_x + 16\,{}_4p_x \ldots + (\omega-x-1)^2\,{}_{\omega-x-1}p_x] \\ &\qquad -[{}_2p_x + 4\,{}_3p_x + 9\,{}_4p_x + \ldots + (\omega-x-2)^2\,{}_{\omega-x-1}p_x] \\ &= p_x + 3\,{}_2p_x + 5\,{}_3p_x + 7\,{}_4p_x + \ldots + \big(2(\omega-x-1)-1\big)\,{}_{\omega-x-1}p_x \\ &= \sum_{k=0}^{\omega-x-2} (2k+1)\,{}_{k+1}p_x \quad \text{or} \quad \sum_{k=1}^{\omega-x-1} (2k-1)\,{}_kp_x\,. \end{aligned}$$

Hence

$$\mathbb{V}[K(x)] = \sum_{k=1}^{\omega-x-1} (2k-1)\, {}_kp_x - \Big[\sum_{k=1}^{\omega-x-1} {}_kp_x \Big]^2 .$$

Remark 1.4 Henceforth, we will drop the argument (x), both in $T(x)$ and $K(x)$, to use T and K instead.

1.6 Fractional Ages and Durations

Consider, for instance, a policy written on the life of a woman age 32.5. Then ${}_{\frac{1}{2}}p_{32\frac{1}{2}}$ will be needed, that is, the probability that she will survive to 33, the next integer age for which l_x is available in the life table. But

$$ {}_{\frac{1}{2}}p_{32\frac{1}{2}} = \frac{l_{33}}{l_{32\frac{1}{2}}} $$

cannot be obtained from the life table.

This will require an approximation for l_{x+t}, when x is an integer but $0 < t < 1$. Here are some methods proposed in Table 3.6.1, p75 of the textbook:

(a) Uniform distribution of deaths (UDD):

$$l_{x+t} = (1-t)\, l_x + t\, l_{x+1}\,, \qquad 0 \le t \le 1\,,$$

that is a linear interpolation in the life table.

(b) Constant force of mortality:

$$l_{x+t} = l_x\, e^{-\mu\, t}\,, \qquad 0 \le t < 1\,,$$

where $\mu = -\ln p_x$. This is a piece-wise exponential mortality.

(c) The method of Gaetano Balducci (1917):

$$\frac{1}{l_{x+t}} = (1-t)\,\frac{1}{l_x} + t\,\frac{1}{l_{x+1}}\,, \qquad 0 \le t \le 1\,,$$

that is a reciprocal linear interpolation.

Theorem 1.1 Let $S = T - K$ be the random fraction of year lived in the year of death. Then under UDD, S and K are independent random variables and S is uniformly distributed, that is $S \sim U[0,1]$.

Proof: For any $0 \le s \le 1$ and $k = 0, 1, 2, \ldots, \omega - x - 1$

$$\begin{aligned}
\mathbb{P}\{S \le s \mid K = k\} &= \frac{\mathbb{P}\{S \le s, K = k\}}{\mathbb{P}\{K = k\}} \\
&= \frac{\mathbb{P}\{k \le T \le k + s\}}{\mathbb{P}\{K = k\}} \\
&= \frac{{}_{k|s}q_x}{{}_{k|}q_x} = \frac{{}_kp_x \; {}_sq_{x+k}}{{}_kp_x \; q_{x+k}} \\
&= \frac{s \, q_{x+k}}{q_{x+k}} \,, \qquad \text{under UDD} \\
&= s \,,
\end{aligned}$$

which is independent of k. This shows that S and K are independent random variables. Then this also means that $\mathbb{P}\{S \le s\} = s$, showing that $S \sim U[0,1]$. □

This simplifying UDD assumption helps in the evaluation of functions of T, reducing them to evaluating the corresponding functions for K. Take for instance the complete expectation of future lifetime:

$$\begin{aligned}
\mathbb{E}(T) &= \mathbb{E}(K) + \mathbb{E}(S) \\
\mathring{e}_x &= e_x + \mathbb{E}(S) \\
&= e_x + \frac{1}{2} \,, \qquad \text{under UDD} \,. \qquad (1.2)
\end{aligned}$$

(1.2) is now the standard approximation for $\mathring{e}_x$ when life table values are available only at integer ages. Similarly,

$$\begin{aligned}
\mathbb{V}(T) &= \mathbb{V}(K) + \mathbb{V}(S) \,, \qquad \text{under UDD, by independence} \\
&= \mathbb{V}(K) + \frac{1}{12} \,.
\end{aligned}$$

In comparing the different approximation methods, Balducci's is more tractable in constructing mortality tables because ${}_{1-t}q_{x+t} = (1-t)\, q_x$ for x integer and $0 < t < 1$. This is more useful than ${}_tq_x = t\, q_x$ (under UDD) in computational formulas. But for evaluations done on the computer, this difference is of no importance, hence UDD is preferred to Balducci's assumption.

1.7 Select and Ultimate Life Tables

A selection takes place when certain insurances are underwritten (e.g. medical examination for large face value policies). This introduces a new variable in the model, apart from age:

$[x]$ = a life age x, at selection,
$[x]+k$ = a life age $x+k$, selected k years ago at age x.

The effect of selection wears off with time. A positive selection during n years implies:

$$q_{[x]} < q_{[x-1]+1} < q_{[x-2]+2} < \cdots < q_{[x-n+1]+n-1} \ .$$

After this *selection period* the difference

$$q_{[x-n]+n} - q_{[x-n+1]+n-1}$$

becomes negligible. Usual selection periods = 3, 5, 10 or 15 years. Note that $l_{[x]+n} = l_{x+n}$ for all $n \geq 3$, the number of selection years.

3–year Select Life Table				**Ultimate**
$[x]$	$l_{[x]}$	$l_{[x]+1}$	$l_{[x]+2}$	l_{x+3}
$[20]$	$l_{[20]}$	$l_{[20]+1}$	$l_{[20]+2}$	l_{23}
$[21]$	$l_{[21]}$	$l_{[21]+1}$	$l_{[21]+2}$	l_{24}
$[22]$	$l_{[22]}$	$l_{[22]+1}$	$l_{[22]+2}$	l_{25}
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$

Table 1.1: Select and Ultimate Life Table

All other life table functions use the same notation:

$$\begin{aligned} d_{[x]+k} &= l_{[x]+k} - l_{[x]+k+1} \ , \\ q_{[x]+k} &= \frac{d_{[x]+k}}{l_{[x]+k}} \ , \\ {}_{j}p_{[x]+k} &= \frac{l_{[x]+k+j}}{l_{[x]+k}} \ \ldots \end{aligned}$$

For instance, for a 3–year Select and Ultimate table, then

$$d_{[20]} = l_{[20]} - l_{[20]+1} \ , \quad d_{[20]+1} = l_{[20]+1} - l_{[20]+2} \ , \text{ but } \quad d_{[20]+2} = l_{[20]+2} - l_{23} \ .$$

Remark 1.5 The ultimate table of a Select and Ultimate Life Table usually shows a better survival experience than that of the general population. This is due to the fact that every life in the group used to calculate the Select and Ultimate Life Table passed, at some point, the selection requirement. Many lives in the general population would never pass it, but are included in the calculation of a general table.

Chapter 2

Life Insurance

2.1 Insurances Payable at Death

2.1.1 Whole Life Insurance

If \$1 is paid at the death of (x) to the designated beneficiary, the present value (pv) of the insurance benefit is $Z_T = v^T$ and

$$\mathbb{E}(Z_T) = \int_0^{\omega - x} v^t \, {}_t p_x \, \mu(x+t) \, dt \ .$$

This expected pv is called an *actuarial pv* (apv) and is denoted $\bar{A}_x = \mathbb{E}(v^T)$. It is also called a *Net Single Premium* (NSP):

Single $\longrightarrow$ only one premium paid at age x (not an annuity),
Net $\longrightarrow$ does not account for expenses or profit.

Also

$$\begin{aligned} \mathbb{E}(Z_T^2) &= \int_0^{\omega - x} \underbrace{v^{2t}}_{e^{-2\delta\, t}} \, {}_t p_x \, \mu(x+t) \, dt \\ &= {}^2\bar{A}_x \ , \qquad \text{i.e. the apv at } \delta' = 2\,\delta \ , \end{aligned}$$

(note that ${}^1\bar{A}_x = \bar{A}_x$). This implies that

$$\mathbb{V}(Z_T) = {}^2\bar{A}_x - \bar{A}_x^2 \ .$$

Remark 2.1 For face values other than \$1, say M, then $\mathbb{E}(M\, v^T) = M\, \bar{A}_x$, $\mathbb{V}(M\, v^T) = M^2\, ({}^2\bar{A}_x - \bar{A}_x^2)$, etc.

Example 2.1 The future lifetime $T(20) \sim \exp(1/60)$, that is it has an exponential distribution with mean 60. Hence $\mu(20+t) = 1/60$ and ${}_tp_{20} = \exp\{-t/60\}$, for $t > 0$.

$$\begin{aligned} \Rightarrow \bar{A}_{20} &= \int_0^\infty e^{-\delta\, t}\, e^{\frac{-t}{60}}\, \frac{1}{60}\, dt = \frac{\frac{1}{60}}{\delta + \frac{1}{60}} = \frac{1}{1 + 60\,\delta} \\ \Rightarrow {}^2\bar{A}_{20} &= \frac{1}{1 + 120\,\delta} \quad \text{and hence} \quad \frac{1}{1+120\,\delta} - \left(\frac{1}{1+60\,\delta}\right)^2 . \end{aligned}$$

See the table below for the values (calculated in Excel) of the moments of $Z_T = 1000\, v^T$ at $\delta = 0.01, 0.02, \ldots, 0.1$.

δ	**1000 $\bar{\mathrm{A}}_{20}$**	**Variance**	**Stand. Dev.**
0.01	625.00	63920.45	252.82
0.02	454.55	87506.08	295.81
0.03	357.14	89840.28	299.73
0.04	294.12	85908.60	293.10
0.05	250.00	80357.14	283.47
0.06	217.39	74692.24	273.30
0.07	192.31	69400.73	263.44
0.08	172.41	64613.11	254.19
0.09	156.25	60331.70	245.63
0.1	142.86	56514.91	237.73

Table 2.1: Whole Life Insurance NSP for (20)

$\triangle$

Other characteristics of the distribution of Z_T could be used to define NSP's for a whole life insurance (although we will study only apv's in this chapter).

Example 2.2 Find the NSP, π, such that:

$$\begin{aligned}
\mathbb{P}\{Z_T \geq \pi\} &\leq 0.25 \, , \\
\Rightarrow \mathbb{P}\{v^T \geq \pi\} &= \mathbb{P}\{T \, \ln v \geq \ln \pi\} \\
&= \mathbb{P}\Big\{T \leq \underbrace{\frac{\ln \pi}{\ln v}}_{t^*}\Big\} \\
&= {}_t q_x \, .
\end{aligned}$$

Find the smallest π such that ${}_t q_x \leq 0.25$ for $t^* = \frac{\ln \pi}{\ln v}$. For instance if $i = 0.06$ and $x = 18$ then $t^* \approx 48$ from the table of ${}_t q_{18}$ values. Hence $\pi \approx 0.06$. Compare this to $\bar{A}_{18} \approx 0.00895$. △

2.1.2 n–Year Term Life Insurance

Benefit of \$1 at death of (x), if it occurs within n years. Here the random pv is given by:

$$Z_T = \left\{ \begin{array}{ll} v^T & \text{if } T \leq n \\ 0 & \text{if } T > n \end{array} \right\} = v^T \, I_{[0,n]}(T) \, .$$

The NSP for a n–year term life insurance is

$$\mathbb{E}(Z_T) = \int_0^n v^t \, {}_t p_x \, \mu(x+t) \, dt := \bar{A}_{\overset{1}{x}:\overline{n}|} \, ,$$

which implies

$$\mathbb{E}(Z_T^2) = \int_0^n v^{2\,t} \, {}_t p_x \, \mu(x+t) \, dt := {}^2\bar{A}_{\overset{1}{x}:\overline{n}|} \, ,$$

and hence

$$\mathbb{V}(Z_T) = {}^2\bar{A}_{\overset{1}{x}:\overline{n}|} - \bar{A}^2_{\overset{1}{x}:\overline{n}|} \, .$$

Example 2.3 If the future lifetime follows De Moivre's distribution, that is $\mu(x) = (\omega - x)^{-1}$ and ${}_t p_x = \frac{\omega-(x+t)}{\omega-x}$, for $0 \leq t \leq \omega - x$ then

$$\begin{aligned}
\bar{A}_{\overset{1}{x}:\overline{n}|} &= \int_0^n \frac{v^t}{(\omega - x)} \, dt \, , \qquad n \leq \omega - x \, , \\
&= \frac{\bar{a}_{\overline{n}|}}{(\omega - x)} \, .
\end{aligned}$$

△

2.1.3 n–Year Pure Endowment

Pays \$1, n–year hence, if (x) is then alive. Here

$$Z_T = \left\{ \begin{array}{ll} 0 & \text{if } T \leq n \\ v^n & \text{if } T > n \end{array} \right\} = v^n \, I_{(n,\infty]}(T) \; .$$

The expected value is

$$\mathbb{E}(Z_T) = v^n \, {}_np_x := A_{x:\overset{\;1}{\overline{n}|}} \; .$$

Remarks 2.1

- other notation ${}_nE_x = A_{x:\overset{\;1}{\overline{n}|}}$,
- no bar over the A because not paid at death.

Similarly

$$\mathbb{E}(Z_T^2) = v^{2\,n} \, {}_np_x := {}^2A_{x:\overset{\;1}{\overline{n}|}} \; ,$$

and hence

$$\mathbb{V}(Z_T) = {}^2A_{x:\overset{\;1}{\overline{n}|}} - (A_{x:\overset{\;1}{\overline{n}|}})^2 = v^{2\,n}\,(1 - {}_np_x)\,{}_np_x = v^{2\,n}\,{}_np_x\,{}_nq_x \; .$$

2.1.4 n–Year Endowment Insurance

Pays \$1 at death of (x), if it occurs within n years, or \$1 at age $x+n$, if (x) survives. Here the random pv is given by:

$$\begin{aligned} Z_T &= \left\{ \begin{array}{ll} v^T & \text{if } T \leq n \\ v^n & \text{if } T > n \end{array} \right. \\ &= \underbrace{v^T \, I_{[0,n]}(T)}_{Z_T^a} + \underbrace{v^n \, I_{(n,\infty]}(T)}_{Z_T^b} \\ &= v^T + (v^n - v^T)\, I_{(n,\infty]}(T) \; . \end{aligned}$$

The expected value is

$$\begin{aligned} \mathbb{E}(Z_T) &= \int_0^n v^t \, {}_tp_x \, \mu(x+t)\, dt + v^n \, {}_np_x \\ &= \bar{A}^{1}_{x:\overline{n}|} + A_{x:\overset{\;1}{\overline{n}|}} := \bar{A}_{x:\overline{n}|} \; . \end{aligned}$$

Similarly

$$\begin{aligned}\mathbb{E}(Z_T^2) &= \int_0^n v^{2\,t}\,{}_tp_x\,\mu(x+t)\,dt + v^{2\,n}\,{}_np_x \\ &= {}^2\bar{A}^{1}_{x:\overline{n}|} + {}^2A_{x:\overline{n}|}^{\;\;\,1} = {}^2\bar{A}_{x:\overline{n}|}\ ,\end{aligned}$$

which implies

$$\mathbb{V}(Z_T) = {}^2\bar{A}_{x:\overline{n}|} - (\bar{A}_{x:\overline{n}|})^2\ .$$

Remark 2.2 Note that here

$$\begin{aligned}Z_T &= Z_T^a && + \quad Z_T^b \\ &= \left\{\begin{array}{l}\text{benefit for a}\\ n\text{-year Term}\end{array}\right\} && + \quad \left\{\begin{array}{l}\text{benefit for a } n\text{-year}\\ \text{Pure Endowment}\end{array}\right\}\ ,\end{aligned}$$

which means that

$$\mathbb{E}(Z_T) = \mathbb{E}(Z_T^a) + \mathbb{E}(Z_T^b)\ .$$

But, on the other hand,

$$\mathbb{V}(Z_T) \neq \mathbb{V}(Z_T^a) + \mathbb{V}(Z_T^b)\ ,$$

because Z_T^a and Z_T^b are independent. In fact,

$$\text{when}\quad Z_T^a = \left\{\begin{array}{ll} v^T & \text{if } T \le n \\ 0 & \text{if } T > n \end{array}\right. \qquad \text{then}\quad Z_T^b = \left\{\begin{array}{ll} 0 & \text{if } T \le n \\ v^n & \text{if } T > n \end{array}\right.\ ,$$

which implies that

$$\begin{aligned}\mathrm{Cov}(Z_T^a, Z_T^b) &= \mathbb{E}(Z_T^a\,Z_T^b) - \mathbb{E}(Z_T^a)\,\mathbb{E}(Z_T^b) \\ &= 0 - \bar{A}^{1}_{x:\overline{n}|}\,A_{x:\overline{n}|}^{\;\;\,1} < 0\ .\end{aligned}$$

2.1.5 m–Year Deferred Whole Life Insurance

Pays \$1 at death of (x), given it occurs more than m years hence:

$$Z_T = \left\{\begin{array}{ll} 0 & \text{if } T \le m \\ v^T & \text{if } T > m \end{array}\right\} = v^T\,I_{(m,\infty]}(T)\ .$$

The expected value is given by

$$\mathbb{E}(Z_T) = \int_m^{\omega-x} v^t \, {}_tp_x \, \mu(x+t) \, dt = {}_{m|}\bar{A}_x \; .$$

Similarly

$$\mathbb{E}(Z_T^2) = \int_m^{\omega-x} v^{2\,t} \, {}_tp_x \, \mu(x+t) \, dt = {}_{m|}{}^2\bar{A}_x \; ,$$

which implies

$$\mathbb{V}(Z_T) = {}_{m|}{}^2\bar{A}_x - ({}_{m|}\bar{A}_x)^2 \; .$$

Example 2.4 Under Gompertz' law, $\mu(x) = Bc^x$, which implies

$$\begin{aligned} \Rightarrow \qquad & {}_tp_x = e^{-\int_x^{x+t} \mu(y) dy} = e^{-\frac{B}{\ln c} \, c^x \, (c^t - 1)} \; , \qquad t > 0 \; , \\ \Rightarrow \qquad & {}_{m|}\bar{A}_x = \int_m^{\omega-x} v^t \, B \, c^{x+t} \, e^{-\frac{B}{\ln c} \, c^x \, (c^t-1)} \, dt \; . \end{aligned}$$

Remarks 2.2

$$\bar{A}_{\overset{1}{x}:\overline{m}|} + {}_{m|}\bar{A}_x = \bar{A}_x \; ,$$

since it is true for the corresponding random variables Z_T.

$$\begin{aligned} {}_{m|}\bar{A}_x &= v^m \, {}_mp_x \int_m^{\omega-x} v^{t-m} \, \frac{{}_tp_x}{{}_mp_x} \, \mu(x+t) \, dt \\ &= v^m \, {}_mp_x \int_m^{\omega-x} v^{t-m} \, {}_{t-m}p_{x+m} \, \mu(x+t) \, dt \\ &= v^m \, {}_mp_x \int_0^{\omega-x-m} v^s \, {}_sp_{x+m} \, \mu(x+m+s) \, ds \\ &= A_{x:\overset{1}{\overline{m}|}} \, \bar{A}_{x+m} \; , \end{aligned}$$

that is, a Pure Endowment of $\bar{A}_{x+m}$. Clearly, here $v^m \, {}_mp_x$ acts as a discounting factor for interest and survivorship.

2.1.6 Varying Benefit Insurance

In practice, the amount of benefit received at death is not \$1 nor constant, but often a function of time. For instance, to account for inflation, the *face value* of the policy can be increased at regular intervals, such as every 5 years.

One general way to describe benefits varying with time is to consider then as functions, b_T, of the time of death T. Here are some simple illustrative examples.

Example 2.5 Continuously Increasing Whole Life Insurance

$$\text{Pays } t \text{ at } T = t \quad \Rightarrow \quad Z_T = T\, v^T \; .$$

Here

$$\mathbb{E}(Z_T) = \int_0^{\omega - x} t\, v^t\, {}_tp_x\, \mu(x+t)\, dt \;:=\; (\bar{I}\bar{A})_x \; .$$

By contrast

$$\mathbb{E}(Z_T^2) = \int_0^{\omega - x} t^2\, v^{2t}\, {}_tp_x\, \mu(x+t)\, dt \neq {}^2(\bar{I}\bar{A})_x \; ,$$

which means that the integral must be calculated separately, not simply as another net single premium.

Example 2.6 Increasing Whole Life Insurance

$$\begin{array}{lll} \text{Pays} & \$1 & \text{at death if it occurs in the 1st year,} \\ \text{Pays} & \$2 & \text{at death if it occurs in the 2nd year,} \\ \quad\vdots & \quad\vdots & \end{array}$$

Here

$$Z_T = \lfloor T + 1 \rfloor\, v^T = (\lfloor T \rfloor + 1)\, v^T = (K+1)\, v^T \; ,$$

hence

$$\begin{aligned} \mathbb{E}(Z_T) \;&=\; \int_0^{\omega - x} \lfloor t + 1 \rfloor\, v^t\, {}_tp_x\, \mu(x+t)\, dt \;:=\; (I\bar{A})_x \\ &=\; \sum_{k=0}^{\omega - x - 1} \int_k^{k+1} (k+1)\, v^t\, {}_tp_x\, \mu(x+t)\, dt \\ &=\; \sum_{k=0}^{\omega - x - 1} (k+1)\, \underbrace{v^k\, {}_kp_x}_{A_{\overset{1}{x}:\overline{k}|}}\, \underbrace{\int_0^1 v^s\, {}_sp_{x+k}\, \mu(x+k+s)\, ds}_{\bar{A}_{\overset{1}{x+k}:\overline{1}|}} \\ &=\; \sum_{k=0}^{\omega - x - 1} (k+1)\, A_{\overset{1}{x}:\overline{k}|}\, \bar{A}_{\overset{1}{x+k}:\overline{1}|} = \sum_{k=0}^{\omega - x - 1} (k+1)\, {}_{k|}\bar{A}_{\overset{1}{x}:\overline{1}|} \; , \end{aligned}$$

which can be interpreted as a sum of vertical “blocks” of varying heights. Alternatively, one can “deconstruct” the sum into:

$$\begin{aligned}(I\bar{A})_x &= \sum_{k=0}^{\omega-x-1} (k+1)\,{}_{k|}\bar{A}^{\,1}_{x:\overline{1}|} \\ &= \sum_{k=0}^{\omega-x-1} \sum_{n=0}^{k} {}_{k|}\bar{A}^{\,1}_{x:\overline{1}|} \\ &= \sum_{n=0}^{\omega-x-1} \sum_{k=n}^{\omega-x-1} \int_k^{k+1} v^t\,{}_tp_x\,\mu(x+t)\,dt \\ &= \sum_{n=0}^{\omega-x-1} \underbrace{\int_n^{\omega-x} v^t\,{}_tp_x\,\mu(x+t)\,dt}_{{}_{n|}\bar{A}_x} \\ &= \sum_{n=0}^{\omega-x-1} {}_{n|}\bar{A}_x \ .\end{aligned}$$

Remark 2.3 $\$1,000\,(I\bar{A})_x$ provides coverage of:

$$\begin{array}{ll} \$1,000 & \text{in the 1st year,} \\ \$2,000 & \text{in the 2nd year,} \\ \vdots & \quad\vdots \end{array}$$

In general $(P-Q)\,\bar{A}_x + Q\,(I\bar{A})_x$ provides an initial coverage of P the first year, that increases by Q a year thereafter.

Example 2.7 m-thly Increasing Whole Life Insurance

$$\begin{array}{lll} \text{Pays} & \$\frac{1}{m} & \text{if} \quad 0 < T \le \dfrac{1}{m} \\ \text{Pays} & \$\frac{2}{m} & \text{if} \quad \dfrac{1}{m} < T \le \dfrac{2}{m} \\ \quad\vdots & & \quad\vdots \end{array}$$

Here $Z_T = \frac{\lfloor m\,T+1 \rfloor}{m}\, v^T$, which implies that

$$\mathbb{E}(Z_T) = \int_0^{\omega-x} \frac{\lfloor m\,t+1 \rfloor}{m}\, v^t\,{}_tp_x\,\mu(x+t)\,dt := (I^{(m)}\bar{A})_x \ .$$

Remarks 2.3 (1) $\lim_{m\to\infty}(I^{(m)}\bar{A})_x = (\bar{I}\bar{A})_x$ is used as an approximation for large values of m (for instance $m = 52$).

(2) $\mathbb{E}(Z_T^2) \neq {}^2(I\bar{A})_x$, not even for $m = 1$.

(3) A similar decomposition of vertical "blocks" is also possible in the fully continuous case:

$$\begin{aligned}
(\bar{I}\bar{A})_x &= \int_0^{\omega-x} t\ \underbrace{v^t\ {}_tp_x\ \mu(x+t)\,dt}_{{}_{t|}\bar{A}^{\,1}_{x:\overline{dt}|}} \\
&= \int_0^{\omega-x} \left(\int_0^t ds\right) v^t\ {}_tp_x\ \mu(x+t)\,dt \\
&= \int_0^{\omega-x}\int_s^{\omega-x} v^t\ {}_tp_x\ \mu(x+t)\,dt\,ds \\
&= \int_0^{\omega-x} v^s\ {}_sp_x \int_s^{\omega-x} v^{t-s}\ {}_{t-s}p_{x+s}\ \mu(x+t)\,dt\,ds \\
&= \int_0^{\omega-x} A_{x:\overline{s}|}^{\;\;1} \underbrace{\int_0^{\omega-x-s} v^u\ {}_up_{x+s}\ \mu(x+s+u)\,du}_{\bar{A}_{x+s}}\ ds \\
&= \int_0^{\omega-x} {}_{s|}\bar{A}_x\,ds\ ,
\end{aligned}$$

which can be seen as a "sum" of horizontal "blocks".

Example 2.8 Decreasing n-year Term Insurance

$$\begin{array}{lll}
\text{Pays} & \$n & \text{if}\quad 0 \le T < 1\ , \\
\text{Pays} & \$n-1 & \text{if}\quad 1 \le T < 2\ , \\
\quad\vdots & & \qquad\vdots \\
\text{Pays} & \$1 & \text{if}\quad n-1 \le T < n\ .
\end{array}$$

Here

$$Z_T = \left\{\begin{array}{ll} \big(n - \lfloor T \rfloor\big)\, v^T & \text{if } T \le n \\ 0 & \text{if } T > n \end{array}\right\} = \left\{\begin{array}{ll} \big(n + 1 - \lfloor T+1 \rfloor\big)\, v^T & \text{if } T \le n \\ 0 & \text{if } T > n \end{array}\right. ,$$

which implies that

$$\mathbb{E}(Z_T) = \int_0^n (n - \lfloor t \rfloor)\, v^t\, {}_tp_x\, \mu(x+t)\, dt \ := \ (D\bar{A})^{1}_{x:\overline{n}|}\,.$$

Exercise 2.1 Prove that:

(a) $(D\bar{A})^{1}_{x:\overline{n}|} = n\,\bar{A}^{1}_{x:\overline{1}|} + (n-1)\,{}_{1|}\bar{A}^{1}_{x:\overline{1}|} + \ldots + {}_{n-1|}\bar{A}^{1}_{x:\overline{1}|} = \sum_{j=1}^{n} j\,{}_{n-j|}\bar{A}^{1}_{x:\overline{1}|}$.

(b) $(D\bar{A})^{1}_{x:\overline{n}|} = \bar{A}^{1}_{x:\overline{n}|} + \bar{A}^{1}_{x:\overline{n-1}|} + \ldots + \bar{A}^{1}_{x:\overline{1}|} = \sum_{j=1}^{n} \bar{A}^{1}_{x:\overline{j}|}$.

(c) $(D\bar{A})^{1}_{x:\overline{n}|} = (n+1)\,\bar{A}^{1}_{x:\overline{n}|} - (I\bar{A})^{1}_{x:\overline{n}|}$.

See Table 4.2.1 in the Actuarial Mathematics textbook for a complete list of the different types of life insurance contracts payable at death.

2.2 Insurances Payable at the End of the Year of Death

In practice death benefits cannot be paid exactly at the time of death of policy holders. Usually there will be some time lag between the time of death $T(x)$, of an insured (x), and the time of payment of the benefit by the insurance company.

If this lag time, say L, is assumed independent from the time of death $T = T(x)$, then the net single premiums calculated in the previous section can be adapted to incorporate the effect of this time lag:

$$E(v^{T+L}) = E(v^T)\, E(v^L) = \bar{A}_x\, M_L(-\delta)\ ,$$

where the time lag distribution of the insurance company is assumed know, so that the moment generating function M_L of L can be computed at $-\delta$.

In practice the distribution of L is not known and it is sometimes difficult to estimate for lack of past data (e.g. new product, change in administration practices, mergers). An approximation is used instead, based on the assumption that deaths are uniformly distributed over the year (UDD) and that lags average to a half–year.

Example 2.9 Whole Life Insurance

If \$1 is paid at the end of the year of death of (x), then the present value (pv) of the insurance benefit is

$$Z_K = v^{K+1}\ , \qquad \text{where } K = \lfloor T \rfloor\ .$$

Here the appropriate distribution is

$$\Pr\{K = k\} = {}_{k|}q_x = {}_kp_x\, q_{x+k}\ , \qquad k = 0, 1, \ldots$$

and hence

$$\mathbb{E}(Z_K) = \sum_{k=0}^{\omega-x-1} v^{k+1}\, {}_{k|}q_x := A_x\ .$$

Note that as for $\bar{A}_x$, here

$$\mathbb{E}(Z_K^2) := {}^2A_x\ ,$$

implying that

$$\mathbb{V}(Z_K) = {}^2A_x - A_x^2\ .$$

Example 2.10 n–Year Term Life Insurance

Here the random pv is given by:

$$Z_K = \begin{cases} v^{K+1} & \text{if } K < n \\ 0 & \text{if } K \geq n \end{cases}\ .$$

The NSP for this n–year term life insurance is thus

$$\mathbb{E}(Z_K) = \sum_{k=0}^{n-1} v^{k+1}\, {}_{k|}q_x := A_{\overset{1}{x}:\overline{n}|}\ .$$

Again variances are easily obtained, since

$$\mathbb{E}(Z_K^2) := {}^2A_{\overset{1}{x}:\overline{n}|}\ ,$$

and hence

$$\mathbb{V}(Z_K) = {}^2A_{\overset{1}{x}:\overline{n}|} - A^2_{\overset{1}{x}:\overline{n}|}\ .$$

Example 2.11 n–Year Endowment Insurance

This time the random pv is given by:

$$Z_K = \left\{ \begin{array}{ll} v^{K+1} & \text{if } K < n \\ v^n & \text{if } K \geq n \end{array} \right\} = Z_K^a + Z_K^b \ ,$$

where Z_K^a represents the random pv of a n–year term life insurance, while Z_K^b is that of a n–year pure endowment. The NSP is thus

$$\mathbb{E}(Z_K) = \sum_{k=0}^{n-1} v^{k+1} \, {}_{k|}q_x + v^n \, {}_n p_x := A_{x:\overline{n}|} = A^{1}_{x:\overline{n}|} + A_{x:\overset{1}{\overline{n}|}} \ .$$

Variances are obtained with:

$$\mathbb{E}(Z_K^2) := {}^2A_{x:\overline{n}|} = {}^2A^{1}_{x:\overline{n}|} + {}^2A_{x:\overset{1}{\overline{n}|}} \ ,$$

and

$$\mathbb{V}(Z_K) = {}^2A_{x:\overline{n}|} - A^2_{x:\overline{n}|} \neq \mathbb{V}(Z_K^a) + \mathbb{V}(Z_K^b) \ .$$

Example 2.12 Increasing n–Year Term Life Insurance

Consider

$$Z_K = \left\{ \begin{array}{ll} (K+1)\, v^{K+1} & \text{if } K < n \\ 0 & \text{if } K \geq n \end{array} \right. \ .$$

Then the NSP is given by

$$\mathbb{E}(Z_K) = \sum_{k=0}^{n-1} (k+1)\, v^{k+1} \, {}_{k|}q_x := (IA)^{1}_{x:\overline{n}|} = \sum_{j=0}^{n-1} {}_{j|}A^{1}_{x:\overline{n-j}|} \ .$$

Example 2.13 Decreasing n-year Term Insurance

Similarly

$$Z_K = \left\{ \begin{array}{ll} (n-K)\, v^{K+1} & \text{if } K < n \\ 0 & \text{if } K \geq n \end{array} \right. \ ,$$

which implies that the NSP is given by

$$\begin{aligned} \mathbb{E}(Z_K) &= \sum_{k=0}^{n-1} (n-k)\, v^{k+1} \, {}_{k|}q_x := (DA)^{1}_{x:\overline{n}|} \\ &= \sum_{j=1}^{n} A^{1}_{x:\overline{j}|} = \sum_{k=1}^{n} k \, {}_{n-k|}A^{1}_{x:\overline{1}|} \\ &= (n+1)\, A^{1}_{x:\overline{n}|} - (IA)^{1}_{x:\overline{n}|} \ . \end{aligned}$$

See Table 4.3.1 for an exhaustive list of these simple insurance contracts with benefits payable at the end of the year of death.

These simple contracts can be combined to construct more complex insurance coverages. The corresponding combinations of NSP's gives a simple way to price these more complex products.

Example 2.14 Give an expression for the NSP of a whole life insurance payable at the end of the year of death of (x) if the coverage is \$10,000 for the next 10 years, \$20,000 for the following 15 and \$25,000 thereafter:

$$10,000\, A_x + 10,000\, {}_{10|}A_x + 5,000\, {}_{25|}A_x \ ,$$

or

$$10,000\, A^{\,1}_{x:\overline{10}|} + 20,000\, {}_{10|}A^{\,1}_{x:\overline{15}|} + 25,000\, {}_{25|}A_x \ ,$$

or equivalently

$$25,000\, A_x - 5,000\, A^{\,1}_{x:\overline{25}|} - 10,000\, A^{\,1}_{x:\overline{10}|} \ .$$

2.3 Relation Between $\bar{A}$ and A

Remember that $T = K + S$, where $K = \lfloor T \rfloor$ and $S \in [0,1]$. In addition, under UDD, K and S are independent and $S \sim U[0,1]$.

Example 2.15 Whole Life Insurance

$$\begin{aligned} \bar{A}_x &= \mathbb{E}(v^T) = \mathbb{E}(v^{K+S}) = \mathbb{E}(v^{K+1}\, v^{S-1}) \\ &= \mathbb{E}(v^{K+1})\, \mathbb{E}(v^{S-1}) \ , \qquad \text{by independence} \ , \\ &= A_x \int_0^1 e^{\delta(1-s)}\, ds \ , \qquad \text{since } S \in U[0,1] \ , \\ &= A_x\, \bar{s}_{\overline{1}|} = A_x \frac{[(1+i)-1]}{\delta} = \frac{i}{\delta} A_x \ . \end{aligned}$$

Interpretation: Note that $\bar{A}_x > A_x$. This is due to the fact that to pay the \$1 at death, instead than at the end of the year at death, the insurance company needs A_x, plus an additional charge of $\frac{(i-\delta)}{\delta} A_x$, to pay for the interest loss in the year of death.

Example 2.16 n–Year Term Life Insurance

Under UDD:

$$\begin{aligned}
\bar{A}^{\,1}_{x:\overline{n}|} &= \mathbb{E}(v^T\, I[T \le n]) \\
&= \mathbb{E}(v^{K+1}\, I[K < n])\, \mathbb{E}(v^{S-1})\,, \qquad \text{since } \Pr\{T = n\} = 0\,, \\
&= A^{\,1}_{x:\overline{n}|} \int_0^1 e^{\delta(1-s)}\, ds\,, \qquad \text{since } S \in U[0,1]\,, \\
&= \frac{i}{\delta}\, A^{\,1}_{x:\overline{n}|}\,.
\end{aligned}$$

Note that $\bar{A}_{x:\overline{n}|} \ne \frac{i}{\delta}\, A_{x:\overline{n}|}$, but rather

$$\bar{A}_{x:\overline{n}|} = \bar{A}^{\,1}_{x:\overline{n}|} + A_{x:\overset{1}{\overline{n}|}} = \frac{i}{\delta}\, A^{\,1}_{x:\overline{n}|} + A_{x:\overset{1}{\overline{n}|}}\,.$$

In fact all linear combinations of $\bar{A}$ values $= i/\delta$ times the same linear combination in A values, as in the following examples.

Example 2.17 Increasing n–Year Term Life Insurance

$$\begin{aligned}
(I\bar{A})^{\,1}_{x:\overline{n}|} &= \sum_{k=0}^{n-1} (k+1)\, {}_{k|}\bar{A}^{\,1}_{x:\overline{1}|} \quad \left(\text{alternatively } = \sum_{j=0}^{n-1} {}_{j|}\bar{A}^{\,1}_{x:\overline{n-j}|}\right) \\
&= \sum_{k=0}^{n-1} (k+1)\, v^k\, {}_kp_x\, \bar{A}_{\overset{1}{x+k}:\overline{1}|} \quad \left(= \sum_{j=0}^{n-1} v^j\, {}_jp_x\, \bar{A}_{\overset{1}{x+j}:\overline{n-j}|}\right) \\
&= \frac{i}{\delta}\, (IA)^{\,1}_{x:\overline{n}|}\,.
\end{aligned}$$

Note that this implies that:

$$\begin{aligned}
(D\bar{A})^{\,1}_{x:\overline{n}|} &= \frac{i}{\delta}\, (DA)^{\,1}_{x:\overline{n}|}\,, \\
(I\bar{A})_x &= \frac{i}{\delta}\, (IA)_x\,,
\end{aligned}$$

but, on the other hand,

$$(\bar{I}\bar{A})_x \ne \frac{i}{\delta}\, (IA)_x\,.$$

Here the appropriate approximation is given by

$$\begin{aligned}(\bar{I}\bar{A})_x &= \mathbb{E}(T\,v^T) = \mathbb{E}[(K+S)\,v^{K+S}] \\ &= \mathbb{E}[(K+1)\,v^{K+1}\,v^{S-1} + (S-1)\,v^{K+1}\,v^{S-1}] \\ &= \frac{i}{\delta}\,(IA)_x + \mathbb{E}[(S-1)\,v^{S-1}]\,\mathbb{E}[v^{K+1}]\;,\end{aligned}$$

where

$$\begin{aligned}\mathbb{E}[(S-1)\,v^{S-1}] &= -\int_0^1 u\,e^{\delta\,u}\,du \\ &= \left.\frac{-u\,e^{\delta\,u}}{\delta}\right|_0^1 + \int_0^1 \frac{e^{\delta\,u}}{\delta}\,du \\ &= \frac{-e^{\delta}}{\delta} + \frac{(e^{\delta}-1)}{\delta^2}\;.\end{aligned}$$

Hence

$$(\bar{I}\bar{A})_x = \frac{i}{\delta}\,(IA)_x - A_x\left[\frac{(1+i)}{\delta} - \frac{i}{\delta^2}\right]\;.$$

2.4 Recursive and Differential Equations

Consider the following examples.

Example 2.18 Discrete Whole Life Insurance

In the next year of life, (x) will either die and trigger a benefit payment, or survive to age $x+1$, keeping a full insurance coverage. Hence:

$$A_x = v\,q_x + v\,p_x\,A_{x+1}\;,\qquad x = 0,1,\ldots,\omega-2\;,$$

where $A_{\omega-1} = v$.

Proof: $A_x = A_{\overset{1}{x}:\overline{n|}} + {}_{n|}A_x$, for any $n \geq 1$. Hence for $n=1$ we have $A_x = A_{\overset{1}{x}:\overline{1|}} + {}_{1|}A_x$, as above. In general, for a n–year term insurance:

$$A_{\overset{1}{x}:\overline{n|}} = v\,q_x + v\,p_x\,A_{\overset{1}{x+1}:\overline{n-1|}}\;.$$

□

Example 2.19 Continuous Whole Life Insurance

The infinitesimal change in NSP, exactly at the attainment of age x, can be measured through:

$$\begin{aligned}\frac{d}{dx}\bar{A}_x &= \bar{A}_x\,[\mu(x)+\delta] \quad - \quad \mu(x)\\ &= \text{increase with interest and survivorship}\\ &\qquad -\text{decrease with mortality}\ .\end{aligned}$$

Proof: By parts

$$\begin{aligned}\bar{A}_x &= \int_0^{\omega-x} v^t{}_tp_x\,\mu(x+t)\,dt\\ &= v^t\,(-{}_tp_x)\big|_0^{\omega-x} - \int_0^{\omega-x}(-\delta\,v^t)\,(-{}_tp_x)\,dt\\ &= 1-\delta\int_0^{\omega-x} v^t\,{}_tp_x\,dt\ . \qquad (2.1)\end{aligned}$$

Now, using Leibnitz' rule we can find the derivative to be:

$$\begin{aligned}\frac{d}{dx}\bar{A}_x &= \frac{d}{dx}\left[1-\delta\int_0^{\omega-x} v^t\,{}_tp_x\,dt\right]\\ &= -\delta\,v^{\omega-x}\,{}_{\omega-x}p_x - \delta\int_0^{\omega-x}\frac{\partial}{\partial x}v^t\,{}_tp_x\,dt\\ &= 0-\delta\int_0^{\omega-x} v^t\,{}_tp_x\,[\mu(x)-\mu(x+t)]\,dt\ .\end{aligned}$$

Substituting (2.1) into this last expression gives:

$$\frac{d}{dx}\bar{A}_x = -\delta\left[\frac{1-\bar{A}_x}{\delta}\right]\mu(x)+\delta\,\bar{A}_x = \bar{A}_x\,[\mu(x)+\delta]-\mu(x)\ .$$

□

Chapter 3

Life Annuities

3.1 Continuous Life Annuities

3.1.1 Whole Life Annuity

$Y = \bar{a}_{\overline{T}|}$ is the *random* present value of an annuity paid continuously until death.

$$\begin{aligned} \bar{a}_x := \mathbb{E}(Y) = \mathbb{E}(\bar{a}_{\overline{T}|}) &= \int_0^{\omega-x} \bar{a}_{\overline{t}|}\, {}_tp_x\, \mu(x+t)\, dt \\ &= -\, \bar{a}_{\overline{t}|}\, {}_tp_x \Big|_0^{\omega-x} + \int_0^{\omega-x} {}_tp_x\, d(\bar{a}_{\overline{t}|}) \ , \end{aligned}$$

where (i) $\bar{a}_{\overline{0}|}\, {}_0p_x = 0$; (ii) $\bar{a}_{\overline{\omega-x}|}\, {}_{\omega-x}p_x = 0$ (even when $\omega = \infty$, since then $\bar{a}_{\overline{\infty}|} = 1/\delta$, for any $\delta > 0$) and (iii) $\bar{a}_{\overline{t}|} = \int_0^t v^s\, ds$, which implies $d(\bar{a}_{\overline{t}|}) = v^t\, dt$. Putting these 3 terms together we get the following apv:

$$\bar{a}_x = \int_0^{\omega-x} v^t\, {}_tp_x\, dt \ .$$

Interpretation:

$$\underbrace{l_x\, \bar{a}_x}_{\substack{\text{actuarial present value}\\ \text{of annuities of \$1 p.a.}\\ \text{paid to } l_x \text{ lives age } x}} \quad = \quad \underbrace{\int_0^{\omega-x} v^t\, l_{x+t}\, dt}_{\substack{\text{payment at } t \text{ to}\\ \text{the } l_{x+t} \text{ lives}\\ \text{then alive}}} \ ,$$

Remark 3.1 (1) If $\delta = 0$ then $Y = T$ and $\mathbb{E}(Y) = \mathring{e}_x$.

(2) Since $A_{\overset{\,1}{x:\overline{t|}}} = v^t\,{}_tp_x$ can be viewed as a discounting factor with interest and survivorship, then

$$\bar{a}_x = \int_0^{\omega-x} v^t\,{}_tp_x\,dt = \int_0^{\omega-x} A_{\overset{\,1}{x:\overline{t|}}}\,dt$$

is seen as the analogue of $\bar{a}_{\overline{n|}} = \int_0^n v^t\,dt$.

(3) From Theory of Interest, for any T:

$$1 = \delta\,\bar{a}_{\overline{T|}} + v^T \ .$$

In particular,

$$\begin{aligned} 1 &= \delta\,\mathbb{E}(\bar{a}_{\overline{T|}}) + \mathbb{E}(v^T) \\ &= \delta\,\bar{a}_x + \bar{A}_x \ . \end{aligned} \tag{3.1}$$

Interpretation: (x) gives \$1 to the company at time 0, in exchange the company pays interest while (x) is alive, plus the \$1 back at death (contingent bond).

(4) From (3.1), for $\delta > 0$:

$$\bar{a}_x = \mathbb{E}(Y) = \frac{1 - \mathbb{E}(Z_T)}{\delta} = \frac{1 - \bar{A}_x}{\delta} \ .$$

Similarly, $\mathbb{V}(Y) = \mathbb{V}(Z_T)/\delta^2$ implies

$$\mathbb{V}(\bar{a}_{\overline{T|}}) = \frac{1}{\delta^2}\,[{}^2\bar{A}_x - \bar{A}_x^2] \ .$$

Exercise: Calculating $\mathbb{E}(Y^2)$, prove the following alternate expression:

$$\mathbb{V}(Y) = \frac{2}{\delta}\,(\bar{a}_x - {}^2\bar{a}_x) - (\bar{a}_x)^2 \ .$$

(5) It is clear from (4), above, that the random variables Y and Z_T are related. In fact:

$$\begin{aligned}
\mathrm{Cov}(Y, Z_T) &= \mathbb{E}(Y\, Z_T) - \mathbb{E}(Y)\,\mathbb{E}(Z_T) \\
&= \mathbb{E}(\bar{a}_{\overline{T|}}\, v^T) - \bar{a}_x\, \bar{A}_x \\
&= \mathbb{E}\left[\frac{(1-v^T)}{\delta}\, v^T\right] - \bar{a}_x\, \bar{A}_x \\
&= \frac{1}{\delta}\,[\bar{A}_x - {}^2\bar{A}_x] - \frac{(1-\bar{A}_x)}{\delta}\,\bar{A}_x \\
&= \frac{[\bar{A}_x^2 - {}^2\bar{A}_x]}{\delta} = -\frac{1}{\delta}\mathbb{V}(v^T)\;.
\end{aligned}$$

3.1.2 Other Annuities

See Table 5.2.1 on page 142 of the textbook for details on the following examples.

Example 3.1 (Temporary Annuity)
Here the pv is given by:

$$Y = \begin{cases} \bar{a}_{\overline{T|}} & \text{if } T < n \\ \bar{a}_{\overline{n|}} & \text{if } T \ge n \end{cases}\;.$$

Show that

$$\bar{a}_{x:\overline{n|}} := \mathbb{E}(Y) = \int_0^n v^t\, {}_tp_x\, dt$$

and

$$\mathbb{V}(Y) = \frac{1}{\delta^2}\,[{}^2\bar{A}_{x:\overline{n|}} - \bar{A}^2_{x:\overline{n|}}]\;.$$

Also $1 = \delta\,\bar{a}_{x:\overline{n|}} + \bar{A}_{x:\overline{n|}}$ implies that

$$\mathbb{V}(Y) = \frac{2}{\delta}\,(\bar{a}_{x:\overline{n|}} - {}^2\bar{a}_{x:\overline{n|}}) - (\bar{a}_{x:\overline{n|}})^2\;.$$

Example 3.2 (Deferred Annuity)
This pv is given by:

$$Y = \begin{cases} 0 & \text{if } T < n \\ v^n\bar{a}_{\overline{T-n|}} & \text{if } T \ge n \end{cases}\;.$$

Show that

$$\begin{aligned} {}_{n|}\bar{a}_x := \mathbb{E}(Y) &= \int_n^{\omega-x} v^t \, {}_tp_x \, dt \\ &= A_{\overset{\;\;1}{x:\overline{n|}}} \int_0^{\omega-x-n} v^s \, {}_sp_{x+n} \, ds \\ &= A_{\overset{\;\;1}{x:\overline{n|}}} \, \bar{a}_{x+n} \end{aligned}$$

and

$$\mathbb{V}(Y) = \frac{2}{\delta} \, v^{2n} \, {}_np_x \left(\bar{a}_{x+n} - {}^2\bar{a}_{x+n} \right) - \left({}_{n|}\bar{a}_x \right)^2 .$$

Example 3.3 (Accumulated Value of an Annuity)
By definition:

$$\begin{aligned} \bar{s}_{x:\overline{n|}} := \frac{1}{A_{\overset{\;\;1}{x:\overline{n|}}}} \, \bar{a}_{x:\overline{n|}} &= \int_0^n \frac{v^t}{v^n} \, \frac{{}_tp_x}{{}_np_x} \, dt \\ &= \int_0^n \frac{dt}{v^{n-t} \, {}_{n-t}p_{x+t}} \\ &= \int_0^n \frac{1}{A_{\overset{\;\;\;\;1}{x+t:\overline{n-t|}}}} dt . \end{aligned}$$

It differs from a stochastic definition, where for a single amount of \$1 at 0, the random accumulated value

$$Y = \begin{cases} \frac{(1+i)^n}{\mathcal{L}(x+n)} \, l_x & \text{if } \mathcal{L}(x+n) > 0 \\ 0 & \text{otherwise} \end{cases} ,$$

which implies that

$$\begin{aligned} \mathbb{E}(Y) &= (1+i)^n \, l_x \, \mathbb{E}\left[\frac{1}{\mathcal{L}(x+n)} \right] && (3.2) \\ &\neq \frac{(1+i)^n \, l_x}{l_{x+n}} = \frac{1}{A_{\overset{\;\;\;\;1}{x+t:\overline{n-t|}}}} . && (3.3) \end{aligned}$$

Clearly (3.3) is used as an approximation for the *true* value (3.2), as the expectation is difficult to obtain. For a study of the accuracy of this approximation and more details on this alternate definition, see for instance Ramsay (1993), Arias (1998) or the references therein.

Example 3.4 (Derivatives)

Using Leibnitz' rule we can find the derivative:

$$\begin{aligned}
\frac{d}{dx}\bar{a}_x &= \frac{d}{dx}\int_0^{\omega-x} v^t\,{}_tp_x\,dt \\
&= -v^{\omega-x}\,{}_{\omega-x}p_x + \int_0^{\omega-x} v^t\,{}_tp_x\,[\mu(x)-\mu(x+t)]\,dt \\
&= \mu(x)\,\bar{a}_x - \bar{A}_x \\
&= \mu(x)\,\bar{a}_x - (1-\delta\,\bar{a}_x) \\
&= [\mu(x)+\delta]\,\bar{a}_x - 1\ . \qquad (3.4)
\end{aligned}$$

Remark 3.2 The derivative in (3.4) implies that:

$$\begin{aligned}
\frac{d}{dx}\bar{A}_x &= \frac{d}{dx}[1-\delta\,\bar{a}_x] \\
&= -\delta\,[\mu(x)\,\bar{a}_x - \bar{A}_x] \\
&= \delta\,\bar{A}_x - \delta\,\mu(x)\left[\frac{1-\bar{A}_x}{\delta}\right] \\
&= [\mu(x)+\delta]\,\bar{A}_x - \mu(x)\ .
\end{aligned}$$

3.2 Discrete Life Annuities

3.2.1 Annuity Due

$Y = \ddot{a}_{\overline{K+1|}}$ is the *random* present value of an annuity paid at the beginning of each year, while (x) survives. Then:

$$\begin{aligned}
\ddot{a}_x &:= \mathbb{E}(Y) = \mathbb{E}(\ddot{a}_{\overline{K+1|}}) = \sum_{k=0}^{\omega-x-1} \ddot{a}_{\overline{k+1|}}\,{}_{k|}q_x \\
&= \sum_{k=0}^{\omega-x-1} \ddot{a}_{\overline{k+1|}}\,({}_kp_x - {}_{k+1}p_x) \\
&= \ddot{a}_{\overline{1|}}\,({}_0p_x - {}_1p_x) + \ddot{a}_{\overline{2|}}\,({}_1p_x - {}_2p_x) + \cdots + \ddot{a}_{\overline{\omega-x|}}\,({}_{\omega-x-1}p_x - {}_{\omega-x}p_x) \\
&= \ddot{a}_{\overline{1|}}\,{}_0p_x + (\ddot{a}_{\overline{2|}} - \ddot{a}_{\overline{1|}})\,{}_1p_x + \cdots + (\ddot{a}_{\overline{\omega-x|}} - \ddot{a}_{\overline{\omega-x-1|}})\,{}_{\omega-x-1}p_x \\
&= \sum_{k=0}^{\omega-x-1} (\ddot{a}_{\overline{k+1|}} - \ddot{a}_{\overline{k|}})\,{}_kp_x\ ,
\end{aligned}$$

where $(\ddot{a}_{\overline{k+1}|} - \ddot{a}_{\overline{k}|}) = v^k$ implies that:

$$\ddot{a}_x = \sum_{k=0}^{\omega-x-1} v^k \, {}_kp_x \ .$$

Interpretation:

(1) $\ddot{a}_x = \sum_{k=0}^{\omega-x-1} A_{x:\overset{1}{\overline{k}|}}$.

(2) $\ddot{a}_x = \mathbb{E}[\ddot{a}_{\overline{K+1}|}] = \mathbb{E}[\left[\frac{1-v^{K+1}}{d}\right] = \frac{1}{d}(1-A_x)$, which implies $1 = d\,\ddot{a}_x + A_x$.

Variance: $\mathbb{V}(\ddot{a}_{\overline{K+1}|}) = \frac{1}{d^2}\,\mathbb{V}(v^{K+1}) = \frac{1}{d^2}\,[{}^2A_x - A_x^2]$. Now note that

$${}^2A_x = 1 - (2d - d^2)\,{}^2\ddot{a}_x \ ,$$

from item (2), above, then implies:

$$\mathbb{V}(Y) = \frac{2}{d}\,(\ddot{a}_x - {}^2\ddot{a}_x) - (\ddot{a}_x^2 - {}^2\ddot{a}_x) \ .$$

Other annuities:

(1) $\ddot{a}_{x:\overline{n}|} := \mathbb{E}(Y) = \sum_{k=0}^{n-1} v^k \, {}_kp_x = \frac{1}{d}(1 - A_{x:\overline{n}|})$, where

$$Y = \begin{cases} \ddot{a}_{\overline{K+1}|} & \text{if } K < n \\ \ddot{a}_{\overline{n}|} & \text{if } K \geq n \end{cases}$$

and $\mathbb{V}(Y) = \frac{1}{d^2}\,[{}^2A_{x:\overline{n}|} - A_{x:\overline{n}|}^2]$.

(2) ${}_{n|}\ddot{a}_x := \sum_{k=n}^{\omega-x-1} v^k \, {}_kp_x = \ddot{a}_x - \ddot{a}_{x:\overline{n}|} = A_{x:\overset{1}{\overline{n}|}}\,\ddot{a}_{x+n}$.

(3) $\ddot{s}_{x:\overline{n}|} := \frac{\ddot{a}_{x:\overline{n}|}}{A_{x:\overset{1}{\overline{n}|}}}$.

3.2.2 Annuities Immediate

Similarly for annuities paid at the end of each year, while (x) survives:

$$\begin{aligned} a_x &:= \mathbb{E}(a_{\overline{K}|}) = \sum_{k=1}^{\omega-x-1} a_{\overline{k}|}\, {}_{k|}q_x \\ &= \sum_{k=1}^{\omega-x-1} a_{\overline{k}|}\,({}_kp_x - {}_{k+1}p_x) \\ &= \sum_{k=1}^{\omega-x-1} (a_{\overline{k+1}|} - a_{\overline{k}|})\, {}_{k+1}p_x + \ddot{a}_{\overline{1}|}\, {}_1p_x = \sum_{k=1}^{\omega-x-1} v^k \, {}_kp_x \ . \end{aligned}$$

Remark 3.3 (1) $a_x = \ddot{a}_x - 1$.

(2) $a_x = \mathbb{E}(a_{\overline{K}|}) = \frac{1-(1+i)\,A_x}{i}$ implies that

$$1 = i\,a_x + (1+i)\,A_x\ .$$

Other annuities:

(1) $a_{x:\overline{n}|} := \sum_{k=1}^{n} v^k\,{}_kp_x = \ddot{a}_{x:\overline{n}|} - 1 + A_{x:\overline{n}|}^{\;\;\;1} = \ddot{a}_{x:\overline{n+1}|} - 1$.

(2) ${}_{n|}a_x := \sum_{k=n+1}^{\omega-x-1} v^k\,{}_kp_x = A_{x:\overline{n}|}^{\;\;\;1}\,a_{x+n} = a_x - a_{x:\overline{n}|}$.

(3) $s_{x:\overline{n}|} := \frac{a_{x:\overline{n}|}}{A_{x:\overline{n}|}^{\;\;\;1}}$.

Additional formulas:

(1)

$$\begin{aligned}
A^{1}_{x:\overline{n}|} &= \mathbb{E}[v^{K+1}\,I(K<n)] \\
&= \mathbb{E}[(v\,\ddot{a}_{\overline{K+1}|} - a_{\overline{K}|})\,I(K<n)] \\
&= v\,\mathbb{E}[\ddot{a}_{\overline{K+1}|}\,I(K<n) + \ddot{a}_{\overline{n}|}\,I(K\geq n)] \\
&\qquad -\mathbb{E}[a_{\overline{K}|}\,I(K<n) + a_{\overline{n}|}\,I(K\geq n)] \\
&= v\,\ddot{a}_{x:\overline{n}|} - a_{x:\overline{n}|}\ .
\end{aligned}$$

Note: taking $n = \omega - x - 1$ shows that $A_x = v\,\ddot{a}_x - a_x$.

(2)

$$\begin{aligned}
A_{x:\overline{n}|} &= A^{1}_{x:\overline{n}|} + A_{x:\overline{n}|}^{\;\;\;1} \\
&= v\,\ddot{a}_{x:\overline{n}|} - a_{x:\overline{n}|} + A_{x:\overline{n}|}^{\;\;\;1} \\
&= v\,\ddot{a}_{x:\overline{n}|} - a_{x:\overline{n-1}|}\ .
\end{aligned}$$

3.3 Annuities with m-thly Payments

Example 3.5 (Whole Life m-thly Annuity Due)

$$\ddot{a}_x^{(m)} = \frac{1}{m}\sum_{j\geq 0} v^{\frac{j}{m}}\;{}_{\frac{j}{m}}p_x\ .$$

Approximations for $\bar{a}_x^{(m)}$:

(1) 3-term Woolhouse Approximation:

$$\ddot{a}_x^{(m)} \approx \ddot{a}_x - \frac{m-1}{2m} - \frac{1}{12}\,\frac{(m^2-1)}{m^2}\,[\mu(x)+\delta]\ .$$

Not used in practice. (See Jordan, 1975, p.45, for a derivation).

(2) 2-term Woolhouse Approximation:

$$\ddot{a}_x^{(m)} \approx \ddot{a}_x - \frac{m-1}{2m}\ .$$

Exact if $v^{x+t}\,l_{x+t}$ is linear in t over the range $[0,1]$; that is $v^{x+t}\,l_{x+t} = (1-t)\,v^x\,l_x + t\,v^{x+1}\,l_{x+1}$ (see Jordan, exercises of Chapter II). This is not equivalent to UDD (i.e. l_{x+t} is linear). Was the standard approximation before 1986.

(3) UDD approximation:

$$\ddot{a}_x^{(m)} \approx \ddot{a}_{\overline{1}|}^{(m)}\,\ddot{a}_x - \frac{[s_{\overline{1}|}^{(m)}-1]}{d^{(m)}}\,A_x := \ddot{a}_{\overline{1}|}^{(m)}\,\ddot{a}_x - \beta(m)\,A_x\ . \tag{3.5}$$

Proposition 3.1 The approximation in (3.5) is exact under UDD.

Proof: Define $A_x^{(m)} := \sum_{k=0}^{m(\omega-x)-1} v^{\frac{(k+1)}{m}}\ {}_{\frac{k}{m}}p_x\ {}_{\frac{1}{m}}q_{x+\frac{k}{m}}$. It can be shown that

$$1 = d^{(m)}\,\ddot{a}_x^{(m)} + A_x^{(m)}\ , \tag{3.6}$$

is the m-thly equivalent of $1 = d\,\ddot{a}_x + A_x$ (note that $\lim_{m\to\infty} A_x^{(m)} = \bar{A}_x$ and $\lim_{m\to\infty}\ddot{a}_x^{(m)} = \bar{a}_x$). Also, under UDD, one can show that

$$A_x^{(m)} = i/i^{(m)}\,A_x\ , \tag{3.7}$$

which is the m-thly equivalent of $\bar{A}_x = i/\delta\, A_x$. Then

$$\begin{aligned} d^{(m)}\,\ddot{a}_x^{(m)} + A_x^{(m)} &= d\,\ddot{a}_x + A_x\ , \qquad \text{by (3.6)}\ , \\ \Rightarrow\ d^{(m)}\,\ddot{a}_x^{(m)} &= d\,\ddot{a}_x + A_x - A_x^{(m)} \\ &= d\,\ddot{a}_x + A_x\left(1 - \frac{i}{i^{(m)}}\right)\ , \qquad \text{by (3.7)}\ , \\ \Rightarrow\ \ddot{a}_x^{(m)} &= \frac{d}{d^{(m)}}\,\ddot{a}_x - \frac{1}{d^{(m)}}\left(\frac{(1+i)-1}{i^{(m)}} - 1\right) A_x \\ &= \ddot{a}_{\overline{1}|}^{(m)}\,\ddot{a}_x - \underbrace{\frac{[s_{\overline{1}|}^{(m)} - 1]}{d^{(m)}}}_{\beta(m)}\, A_x \end{aligned}$$

□

Equivalently (using $A_x = 1 - d\,\ddot{a}_x$)

$$\begin{aligned} \ddot{a}_x^{(m)} &= s_{\overline{1}|}^{(m)}\,\ddot{a}_{\overline{1}|}^{(m)}\,\ddot{a}_x - \frac{[s_{\overline{1}|}^{(m)} - 1]}{d^{(m)}} \\ &:= \alpha(m)\,\ddot{a}_x - \beta(m)\ , \end{aligned} \tag{3.8}$$

which is also exact under UDD and is the usual form of the approximation.

Mnemonic:

$$\text{read}\qquad \alpha(m) = \frac{i\,d}{i^{(m)}\,d^{(m)}} \qquad \text{as} \qquad \frac{\text{id}}{\text{I'm damned}}\ ,$$

$$\text{and}\qquad \beta(m) = \frac{i - i^{(m)}}{i^{(m)}\,d^{(m)}} \qquad \text{as} \qquad \frac{\text{I, I'm}}{\text{I'm damned}}\ ,$$

stammering.

For other annuities, just write them as linear combinations of $\ddot{a}_y^{(m)}$ values, for different ages y.

Example 3.6 (Whole Life m-thly Annuity Immediate)

$$a_x^{(m)} = \ddot{a}_x^{(m)} - \frac{1}{m}\ .$$

Approximations for Continuous Annuities:

(2) 2-term Woolhouse Approximation:

$$\ddot{a}_x^{(m)} \approx \ddot{a}_x - \frac{m-1}{2m}$$

and $\lim_{m\to\infty} \ddot{a}_x^{(m)} = \bar{a}_x$ imply that:

$$\bar{a}_x \approx \ddot{a}_x - \frac{1}{2} \,.$$

This is often used in practice.

(3) UDD approximation:

$$\bar{a}_x \approx \alpha(\infty)\, \ddot{a}_x - \beta(\infty) = \frac{i\,d}{\delta^2}\, \ddot{a}_x - \frac{i-\delta}{\delta^2} \,.$$

This one is exact under UDD.

3.4 Recursive Formulas for Life Annuities

For any $n, m \geq 1$:

$$\ddot{a}_x^{(m)} = \ddot{a}_{x:\overline{n}|}^{(m)} + A_{\underset{x:\overline{n}|}{\ \ 1}}\, \ddot{a}_{x+n}^{(m)} \,.$$

In particular, for $m = 1$, that is annual payments, and $n = 1$ we obtain a one–year recursive formula:

$$\ddot{a}_x = 1 + v\, p_x\, \ddot{a}_{x+1} \,.$$

Keeping $n = 1$ but allowing for m-thly payments, $m \geq 1$, we obtain a more complicated recursion:

$$\begin{aligned} \ddot{a}_x^{(m)} &= \ddot{a}_{x:\overline{1}|}^{(m)} + v\, p_x\, \ddot{a}_{x+1}^{(m)} \\ &\approx \ddot{a}_{\overline{1}|}^{(m)} \ddot{a}_{x:\overline{1}|} - \beta(m)\, \underbrace{A_{\overset{1}{x}:\overline{1}|}}_{v\, q_x} + v\, p_x\, \ddot{a}_{x+1}^{(m)} \,, \end{aligned}$$

which is exact under UDD.

3.5 Special Annuities

3.5.1 Complete Annuities Immediate

$\mathring{a}_x^{(1)}$ = the actuarial present value of \$1 p.a., payable to (x) at year–ends and contingent on survival + an adjustment to account for the period survived between the last \$1 payment and T, i.e. S.

In practice, the adjustment is a fraction of the \$1 installment equal to $(T-K)=S$:

$$\begin{aligned} \Rightarrow Y &= a_{\overline{K}|} + S\,v^T \;, \\ \Rightarrow \mathbb{E}(Y) &= \mathbb{E}[a_{\overline{K}|}] + \mathbb{E}(S\,v^{K+S}) \\ &= a_x + \mathbb{E}(S\,v^{S-1})\,A_x \;, \qquad \text{under UDD} \;. \end{aligned}$$

For computational convenience assume that the adjustment is

$$\frac{\bar{s}_{\overline{S}|}}{\bar{s}_{\overline{1}|}} = \frac{\int_0^S e^{\delta\,t}\,dt}{\int_0^1 e^{\delta\,t}\,dt} \approx S \;, \qquad \text{for } S \text{ small} \;.$$

For example:

$$\frac{\bar{s}_{\overline{\frac{1}{2}}|}}{\bar{s}_{\overline{1}|}} = 0.48808 \approx 0.5 \;, \qquad \text{for } i = 10\% \;.$$

This simplifies present value formulas:

$$\begin{aligned} Y &= a_{\overline{K}|} + v^T\,\frac{\bar{s}_{\overline{S}|}}{\bar{s}_{\overline{1}|}} \\ &= \frac{1-v^K}{i} + v^T\left[\frac{(1+i)^S-1}{i}\right] \\ &= \frac{1-v^T}{i} \\ &= \frac{\delta}{i}\,\bar{a}_{\overline{T}|} \;. \end{aligned}$$

Hence

$$\mathring{a}_x^{(1)} = \frac{\delta}{i}\,\mathbb{E}(\bar{a}_{\overline{T}|}) = \frac{\delta}{i}\,\bar{a}_x \;.$$

Similarly we can show that for $m \geq 1$

$$\mathring{a}_x^{(m)} = \frac{\delta}{i^{(m)}}\,\bar{a}_x \;,$$

where

$$Y = a^{(m)}_{\overline{J}|} + v^T \, \frac{1}{m} \, \frac{\bar{s}_{\overline{T-J}|}}{\bar{s}_{\overline{\frac{1}{m}}|}} \ ,$$

and $J = \frac{\lfloor mT \rfloor}{m}$.

3.5.2 Apportionable Life Annuities Due

$\mathring{\ddot{a}}^{\{1\}}_x$ = the actuarial present value of \$1 p.a., payable to (x) at the beginning of each year − a refund to the *payor* (insurer).

In practice, the refund is $(1-S)\,1$, the fraction of the last \$1 unearned by not surviving until $K+1$. It implies that

$$\begin{aligned} \mathbb{E}(Y) &= \mathbb{E}[\ddot{a}_{\overline{K+1}|} - (1-S)\,v^T] \\ &= \ddot{a}_x - \mathbb{E}[(1-S)\,v^{S-1}]\,A_x \ , \qquad \text{under UDD .} \end{aligned}$$

A simplifying assumption is to let the refund equal

$$\frac{\bar{a}_{\overline{1-S}|}}{\bar{a}_{\overline{1}|}} = \frac{\int_0^{1-S} v^u\,du}{\int_0^1 v^u\,du} \approx 1 - S \ , \qquad \text{for } 1-S \text{ small .}$$

This simplifies present value formulas:

$$\begin{aligned} Y &= \ddot{a}_{\overline{K+1}|} - v^T \, \frac{\bar{a}_{\overline{1-S}|}}{\bar{a}_{\overline{1}|}} \\ &= \frac{1-v^{K+1}}{d} - v^T \left[\frac{1-v^{1-S}}{1-v}\right] \\ &= \frac{1-v^T}{d} \\ &= \frac{\delta}{d}\,\bar{a}_{\overline{T}|} \ . \end{aligned}$$

Hence

$$\mathring{\ddot{a}}^{\{1\}}_x = \frac{\delta}{d}\,\bar{a}_x \ ,$$

and in general, for $m \geq 1$

$$\mathring{\ddot{a}}^{\{m\}}_x = \frac{\delta}{d^{(m)}}\,\bar{a}_x \ .$$

Chapter 4

Net Premiums

4.1 Equivalence Principle

When premiums are not single lump sum payments, the present value of premiums received becomes a random variable.

The *net level premium* must be "equivalent" to the NSP $\bar{A}_x$. Let the pv of premiums received satisfy:

$$\mathbb{E}[\text{pv of premiums received}] = \mathbb{E}[\text{pv of benefits paid}] \ .$$

This is called the *equivalence principle*.

Example 4.1 (25) purchases a \$10,000 whole life (continuous) insurance by paying premiums of Π per year (continuously), while alive:

$$\begin{aligned} Z &= \text{pv of benefits paid} = \$10,000\, v^T \ , \\ Y &= \text{pv of premiums received} = \Pi\, \bar{a}_{\overline{T|}} \ . \end{aligned}$$

By the equivalence principle, here:

$$\mathbb{E}(Y) = \mathbb{E}(Z) \quad \Rightarrow \quad \Pi\, \bar{a}_{25} = 10000\, \bar{A}_{25} \ .$$

It implies that the appropriate premium is then

$$\Pi = \frac{10000\, \bar{A}_{25}}{\bar{a}_{25}} \ .$$

Note that matching expected costs under NSP's and instalment premiums does not match the variance of these costs. This implies an increased risk under the instalment premium strategy. For instance, using the Illustrative Life Table, $\$10,000\,\bar{A}_{25} = 840.75$ and $\Pi = 53.49$ at $i = 6\%$, then:

$$\begin{aligned} L_{NSP} &= \$10,000\,v^T - 10000\,\bar{A}_{25}\ , \\ L_{\Pi} &= \$10,000\,v^T - \Pi\,\bar{a}_{\overline{T}|}\ , \end{aligned}$$

which in turn implies,

$$\mathbb{V}(L_{\Pi}) = \left(\$10,000 + \frac{\Pi}{\delta}\right)^2 \mathbb{V}(v^T) > (\$10,000)^2\,\mathbb{V}(v^T) = \mathbb{V}(L_{NSP})\ ,$$

where $\mathbb{V}(v^T) = ({}^2\bar{A}_{25} - \bar{A}_{25}^2)$.

Various Benefit/Premium schemes exist in practice:

- Fully Continuous: benefits paid continuously (i.e. at T for insurances, continuously for annuities) and premiums are paid continuously.
- Fully Discrete: benefits paid on a discrete basis (i.e. at $K + 1$ for insurances, discrete payments for annuities) and discrete premiums (at the beginning of periods).
- Mixed Type: all other (marketable) combinations.

4.2 Fully Continuous Premiums

4.2.1 Whole Life Insurance

Define

$$\begin{aligned} L(T) &= Z_T - \mathcal{P}\,Y = v^T - \mathcal{P}\,\bar{a}_{\overline{T}|} \\ &= \text{pv of the loss to the insurer (a r.v.)}, \\ &= \text{the loss random variable}, \end{aligned}$$

and obtain $\mathcal{P}$ such that $\mathbb{E}(L) = 0$ (that is, by the equivalence principle). Then

$$\mathbb{E}(Z_T) = \mathcal{P}\,\mathbb{E}(Y)\ ,$$

implies that

$$\mathcal{P} = \frac{\bar{A}_x}{\bar{a}_x} = \bar{P}(\bar{A}_x)\ .$$

In $\bar{P}(\bar{A}_x)$, the bar over P illustrates the fact that premiums are paid continuously. The $(\bar{A}_x)$ defines the type of benefit sold.

From Table 6.2.1 (Bowers et al., 1997, page 173) we see that the equivalence principle leads to the following fully continuous premiums:

Benefit	Z_T	Y	$\mathcal{P} = \frac{\mathbb{E}(Z_T)}{\mathbb{E}(Y)}$
1. Whole Life Insurance	v^T	$\bar{a}_{\overline{T}\rceil}$	$\bar{P}(\bar{A}_x) = \frac{A_x}{\bar{a}_x}$
2. n-Year Term Insurance	$\begin{cases} v^T \\ 0 \end{cases}$	$\begin{matrix} \bar{a}_{\overline{T}\rceil} & \text{if } T \le n \\ \bar{a}_{\overline{n}\rceil} & \text{if } T > n \end{matrix}$	$\bar{P}(\bar{A}^{\,1}_{x:\overline{n}\rceil}) = \frac{\bar{A}^{\,1}_{x:\overline{n}\rceil}}{\bar{a}_{x:\overline{n}\rceil}}$
3. n-Year Endowment Ins.	$\begin{cases} v^T \\ v^n \end{cases}$	$\begin{matrix} \bar{a}_{\overline{T}\rceil} & \text{if } T \le n \\ \bar{a}_{\overline{n}\rceil} & \text{if } T > n \end{matrix}$	$\bar{P}(\bar{A}_{x:\overline{n}\rceil}) = \frac{\bar{A}_{x:\overline{n}\rceil}}{\bar{a}_{x:\overline{n}\rceil}}$

Table 4.1: Fully Continuous Premiums

In all cases above, the premiums are paid for the duration of the insurance benefit. When the payment period is <u>shorter</u> than the duration of the contract, this is specified in the symbol.

Example 4.2 (35) purchases a $10,000 20-year endowment (continuous) insurance by (continuous) annual premiums of $\mathcal{P}$, payable while alive, for 10 years. By the equivalence principle:

$$10000\,\bar{A}_{35:\overline{20}\rceil} = \mathcal{P}\,\bar{a}_{35:\overline{10}\rceil}\ ,$$

which implies that

$$\mathcal{P} = \frac{10000\,\bar{A}_{35:\overline{20}\rceil}}{\bar{a}_{35:\overline{10}\rceil}} := 10000\,{}_{10}\bar{P}(\bar{A}_{35:\overline{20}\rceil})\ .$$

Table 6.2.1 also lists premiums paid on a shorter term than the contract:

Benefit	Z_T	Y	$\mathcal{P} = \frac{\mathbb{E}(Z_T)}{\mathbb{E}(Y)}$
4. h-Payment Whole Life	$\left\{\begin{array}{l} v^T \\ v^T \end{array}\right.$	$\bar{a}_{\overline{T}|}$ if $T \le h$ $\bar{a}_{\overline{h}|}$ if $T > h$	${}_h\bar{P}(\bar{A}_x) = \frac{\bar{A}_x}{\bar{a}_{x:\overline{h}|}}$
5. h-Payment, n-Year End.	$\left\{\begin{array}{l} v^T \\ v^T \\ v^n \end{array}\right.$	$\bar{a}_{\overline{T}|}$ if $T \le h$ $\bar{a}_{\overline{h}|}$ if $h < T \le n$ $\bar{a}_{\overline{h}|}$ if $T > n$	${}_h\bar{P}(\bar{A}_{x:\overline{n}|}) = \frac{\bar{A}_{x:\overline{n}|}}{\bar{a}_{x:\overline{h}|}}$

Table 4.2: Fully Continuous Premiums (... continued)

When the benefit is a deferred annuity, the premiums are paid during the deferment period:

Benefit	Z_T	Y	$\mathcal{P} = \frac{\mathbb{E}(Z_T)}{\mathbb{E}(Y)}$
6. n-Year Deferred Annuity	$\left\{\begin{array}{l} 0 \\ v^n\,\bar{a}_{\overline{T-n}|} \end{array}\right.$	$\bar{a}_{\overline{T}|}$ if $T \le n$ $\bar{a}_{\overline{n}|}$ if $T > n$	$\bar{P}({}_{n|}\bar{a}_x) = \frac{{}_{n|}\bar{a}_x}{\bar{a}_{x:\overline{n}|}}$

Table 4.3: Fully Continuous Premiums (... continued)

Other Formulas:

$$\begin{aligned} \bar{P}(\bar{A}_{x:\overline{n}|}) &= \frac{1-\delta\,\bar{a}_{x:\overline{n}|}}{\bar{a}_{x:\overline{n}|}} = \frac{1}{\bar{a}_{x:\overline{n}|}} - \delta\,, \qquad \text{(also for } n \to \infty\text{)} \\ &= \frac{\delta\,\bar{A}_{x:\overline{n}|}}{1-\bar{A}_{x:\overline{n}|}}\,. \end{aligned}$$

Also

$$L = v^T - \mathcal{P}\,\bar{a}_{\overline{T}|} \quad \Rightarrow \quad \begin{aligned} \mathbb{V}(L) &= \mathbb{V}[v^T\,(1+\frac{\mathcal{P}}{\delta}) - \frac{\mathcal{P}}{\delta}] \\ &= (1+\frac{\mathcal{P}}{\delta})^2\,[{}^2\bar{A}_x - \bar{A}_x^2]\,, \end{aligned}$$

for any $\mathcal{P}$. If, in particular, $\mathcal{P} = \bar{P}(\bar{A}_x)$, then

$$\mathbb{V}(L) = \frac{({}^2\bar{A}_x - \bar{A}_x^2)}{(\delta\,\bar{a}_x)^2}\,.$$

4.3 Non–Continuous Premiums

4.3.1 Fully Discrete Premiums

Here it is Table 6.3.1 (Bowers et al., 1997, page 183) that summarizes the different possible premiums.

Numerators are now A_x, $A_{\stackrel{1}{x:\overline{n}|}}$ or $A_{x:\overline{n}|}$, respectively, for cases 1., 2., and 3. in Table 4.1, above. The corresponding denominators are $\ddot{a}_x$, $\ddot{a}_{x:\overline{n}|}$ and again $\ddot{a}_{x:\overline{n}|}$.

The notation is simpler:

$$P_x = \frac{A_x}{\ddot{a}_x}\ , \quad P_{\stackrel{1}{x:\overline{n}|}} = \frac{A_{\stackrel{1}{x:\overline{n}|}}}{\ddot{a}_{x:\overline{n}|}} \quad \text{and} \quad P_{x:\overline{n}|} = \frac{A_{x:\overline{n}|}}{\ddot{a}_{x:\overline{n}|}}\ .$$

For h-payment insurances (cases 4. and 5.), including Pure Endowments:

$${}_hP_x = \frac{A_x}{\ddot{a}_{x:\overline{h}|}}\ , \quad {}_hP_{x:\overline{n}|} = \frac{A_{x:\overline{n}|}}{\ddot{a}_{x:\overline{h}|}} \quad \text{and} \quad {}_hP_{x:\stackrel{\ 1}{\overline{n}|}} = \frac{A_{x:\stackrel{\ 1}{\overline{n}|}}}{\ddot{a}_{x:\overline{h}|}}\ .$$

Note that ${}_hP_{x:\overline{n}|} = {}_hP_{\stackrel{1}{x:\overline{n}|}} + {}_hP_{x:\stackrel{\ 1}{\overline{n}|}}$, for $h \leq n$, although $A_{\stackrel{1}{x:\overline{n}|}}$ is not usually sold with $h < n$.

Finally for deferred annuities (case 6.):

$$P({}_{n|}\ddot{a}_x) = \frac{{}_{n|}\ddot{a}_x}{\ddot{a}_{x:\overline{n}|}}\ .$$

Other Formulas:

$$\begin{aligned} P_{x:\overline{n}|} &= \frac{A_{x:\overline{n}|}}{\ddot{a}_{x:\overline{n}|}} = \frac{1 - d\,\ddot{a}_{x:\overline{n}|}}{\ddot{a}_{x:\overline{n}|}} = \frac{1}{\ddot{a}_{x:\overline{n}|}} - d\ , \qquad \text{(also for } n \to \infty\text{)} \\ &= \frac{d\,A_{x:\overline{n}|}}{1 - A_{x:\overline{n}|}}\ . \end{aligned}$$

Also

$$\begin{aligned} L = v^{K+1} - \mathcal{P}\,\ddot{a}_{\overline{K+1}|} \quad \Rightarrow \quad \mathbb{V}(L) &= \mathbb{V}[v^{K+1}\,(1 + \frac{\mathcal{P}}{d}) - \frac{\mathcal{P}}{d}] \\ &= (1 + \frac{\mathcal{P}}{d})^2\,[{}^2A_x - A_x^2]\ , \end{aligned}$$

for any $\mathcal{P}$. If, in particular, $\mathcal{P} = P_x$, then

$$\mathbb{V}(L) = \frac{({}^2A_x - A_x^2)}{(d\,\ddot{a}_x)^2}\ .$$

4.4 Semi–Continuous Premiums

Here insurances are payable at the moment of death but premiums payable anually. For instance:

$$P(\bar{A}_x) = \frac{\bar{A}_x}{\ddot{a}_x} \,, \quad P(\bar{A}^{1}_{x:\overline{n}|}) = \frac{\bar{A}^{1}_{x:\overline{n}|}}{\ddot{a}_{x:\overline{n}|}} \quad \text{and} \quad P(\bar{A}_{x:\overline{n}|}) = \frac{\bar{A}_{x:\overline{n}|}}{\ddot{a}_{x:\overline{n}|}} \,.$$

These are as in Table 6.2.1 (Bowers et al., 1997) but with denominators in $\ddot{a}$'s rather than $\bar{a}$'s.
Additional Formulas:

$${}_hP(\bar{A}_{x:\overline{n}|}) = \frac{\bar{A}_{x:\overline{n}|}}{\ddot{a}_{x:\overline{h}|}} \approx \frac{\frac{i}{\delta}\, A^{1}_{x:\overline{n}|} + A_{x:\overline{n}|}^{\ \ \ \ 1}}{\ddot{a}_{x:\overline{h}|}} = \frac{i}{\delta}\, {}_hP^{1}_{x:\overline{n}|} + {}_hP_{x:\overline{n}|}^{\ \ \ \ 1} \,.$$

Also

$$L = v^T - \mathcal{P}\, \ddot{a}_{\overline{K+1}|} \,,$$

then implies that

$$\mathbb{V}(L) = \mathbb{V}(v^T) + \mathcal{P}^2\, \mathbb{V}(\ddot{a}_{\overline{K+1}|}) - 2\,\mathcal{P}\,\mathrm{Cov}(v^T, \ddot{a}_{\overline{K+1}|}) \,.$$

4.5 True m-thly Premiums

Here insurances are payable at the end of the year of death but premiums are payable m-thly (no adjustments in the death benefit). For example:

$$P^{(12)}_x = \frac{A_x}{\ddot{a}^{(12)}_x} \,, \quad P^{(12)}_{\substack{1\\ x:\overline{n}|}} = \frac{A^{1}_{x:\overline{n}|}}{\ddot{a}^{(12)}_{x:\overline{n}|}} \quad \text{and} \quad P^{(12)}_{x:\overline{n}|} = \frac{A_{x:\overline{n}|}}{\ddot{a}^{(12)}_{x:\overline{n}|}} \,.$$

Similar formulas can be obtained for $P^{(m)}(\bar{A}_x)$, $P^{(m)}(\bar{A}^{1}_{x:\overline{n}|})$ and $P^{(m)}(\bar{A}_{x:\overline{n}|})$ (for details see Table 6.4.1, Bowers et al., 1997, page 189).

Remark 4.1 (1) The annual premium is $P^{(m)}$, but the amount of each installment is $\frac{1}{m}\,P^{(m)}$.

(2) Use standard approximations [i.e. $\alpha(m)$ and $\beta(m)$].

4.6 Apportionable m-thly Premiums

A portion of the premium is refunded at death, depending on the length of time between death and the time of the next scheduled payment. The refund amount is:

$$\mathcal{P}\,(1-S) \approx \mathcal{P}\,\frac{\bar{a}_{\overline{1-S}|}}{\bar{a}_{\overline{1}|}}\ .$$

For example, an h-payment n-year endowment insurance:

$${}_hP^{\{m\}}(\bar{A}_{x:\overline{n}|})\,\ddot{a}^{\{m\}}_{x:\overline{n}|} = \bar{A}_{x:\overline{n}|}\ ,$$

which implies that

$${}_hP^{\{m\}}(\bar{A}_{x:\overline{n}|}) = \frac{\bar{A}_{x:\overline{n}|}}{\ddot{a}^{\{m\}}_{x:\overline{h}|}} \approx \frac{\bar{A}_{x:\overline{n}|}}{\frac{\delta}{d^{(m)}}\,\bar{a}_{x:\overline{h}|}} = \frac{d^{(m)}}{\delta}\,{}_h\bar{P}(\bar{A}_{x:\overline{n}|})\ .$$

This is different from the approximation in Jordan (1975).

Chapter 5

Net Premiums Reserves

5.1 The Nature of the Reserve

Consider a whole life insurance to (x), payable at the end of the year of death. At issue

$$A_x = P_x\, \ddot{a}_x \quad \Rightarrow \quad \mathbb{E}(L) = A_x - P_x\, \ddot{a}_x = 0\ .$$

Now, $t \geq 0$ years into the contract, if (x) is still alive, then the actuarial present value (apv) of the obligation of the insurer is

$$A_{x+t}\ ,$$

while the apv of future premiums to be received (say t is an integer and we stand just before the next premium payment) is:

$$P_x\, \ddot{a}_{x+t}\ .$$

Hence, the difference:

$$\begin{aligned} \text{reserve} &= A_{x+t} - P_x\, \ddot{a}_{x+t} \\ &= \text{net obligation of the insurer}\ . \end{aligned}$$

Note that usually $A_{x+t} \geq A_x$, while $\ddot{a}_{x+t} \leq \ddot{a}_x$. Hence

$$P_{x+t} = \frac{A_{x+t}}{\ddot{a}_{x+t}} \geq P_x = \frac{A_x}{\ddot{a}_x}\ ,$$

which in turn implies that

$$\text{reserve} = A_{x+t} - P_x\,\ddot{a}_{x+t} \geq A_{x+t} - P_{x+t}\,\ddot{a}_{x+t} = 0\ ,$$

with equality only occurring if $t = 0$.

This shows that keeping premiums constant at P_x per year, instead of increasing them with age to reach P_{x+t} in t years, creates the need for a reserve. The latter is calculated under the same mortality and interest assumptions as those used in calculating premiums.

5.2 Prospective Formulas

In this approach simply discount back to time 0 all *future* obligations and premium income.

Example 5.1 (Whole Life Insurance)

Here the reserve is calculated as:

$${}_tV_x := A_{x+t} - P_x\,\ddot{a}_{x+t}\ ,$$

where in the symbol ${}_tV_x$, the index x stands for the age at issue, while t stands for the time elapsed since issue.

Example 5.2 (h-Payment Whole Life Insurance)

The reserve here is calculated differently if t is during the premium paying period, than if it is passed h:

$${}^h_tV_x := \begin{cases} A_{x+t} - {}_hP_x\,\ddot{a}_{x+t:\overline{h-t}|} & \text{if } t < h \\ A_{x+t} & \text{if } t \geq h \end{cases}\ ,$$

where in the symbol ${}^h_tV$, the super–script h stands for the length of the premium paying period.

Example 5.3 (n-Year, h-Payment Continuous Endowment Insurance)

This situation is even more complicated. The reserve is still calculated by discounting all future cash–flows as:

$${}^h_tV(\bar{A}_{x:\overline{n}|}) := \begin{cases} \bar{A}_{x+t:\overline{n-t}|} - {}_hP(\bar{A}_{x:\overline{n}|})\,\ddot{a}_{x+t:\overline{h-t}|} & \text{if } t < h \\ \bar{A}_{x+t:\overline{n-t}|} & \text{if } h \leq t \leq n \\ 0 & \text{if } t > n \end{cases}\ .$$

Example 5.4 (n-Year Deferred Annuity)

Here the apv of the benefit is also given by an annuity:

$$ {}_tV({}_{n|}\ddot{a}_x) := \begin{cases} {}_{n-t|}\ddot{a}_{x+t} - P({}_{n|}\ddot{a}_x)\,\ddot{a}_{x+t:\overline{n-t}|} & \text{if } t < n \\ \ddot{a}_{x+t} & \text{if } t \geq n \end{cases} . $$

Note that the premium paying period is, as usual, equal to the deferral period, unless otherwise specified.

See Table 7.4.1 (Bowers et al., 1997, page 216) for a complete list of fully discrete reserve formulas.

5.3 Retrospective Formulas

Rather than discounting future cash–flows to quantify the deficiency in future premiums, alternatively one can accumulate the past premium excess.

Consider the accumulated value to age $x+t$ of all premiums received as well as the cost of insurance:

$$ P_x\,\ddot{s}_{x:\overline{t}|} - \frac{A_{\overset{1}{x}:\overline{t}|}}{A_{x:\overset{\,1}{\overline{t}|}}} . $$

This should, at any time t cancel out all future deficiencies in premiums.

Proposition 5.1

$$ {}_tV_x = P_x\,\ddot{s}_{x:\overline{t}|} - \frac{A_{\overset{1}{x}:\overline{t}|}}{A_{x:\overset{\,1}{\overline{t}|}}} . $$

Proof: By definition of the prospective reserve

$$ \begin{aligned} {}_tV_x &= A_{x+t} - P_x\,\ddot{a}_{x+t} + \frac{\overbrace{A_x - P_x\,\ddot{a}_x}^{0}}{A_{x:\overset{\,1}{\overline{t}|}}} \\ &= P_x \left[\frac{\ddot{a}_x - A_{x:\overset{\,1}{\overline{t}|}}\,\ddot{a}_{x+t}}{A_{x:\overset{\,1}{\overline{t}|}}} \right] - \left[\frac{A_x - A_{x:\overset{\,1}{\overline{t}|}}\,A_{x+t}}{A_{x:\overset{\,1}{\overline{t}|}}} \right] \\ &= P_x\,\frac{\ddot{a}_{x:\overline{t}|}}{A_{x:\overset{\,1}{\overline{t}|}}} - \frac{A_{\overset{1}{x}:\overline{t}|}}{A_{x:\overset{\,1}{\overline{t}|}}} = P_x\,\ddot{s}_{x:\overline{t}|} - {}_tk_x . \end{aligned} $$

□ Note the new symbol

$${}_tk_x = \frac{A^{1}_{x:\overline{t|}}}{A_{x:\overline{t|}}^{\;\;1}} \,,$$

for the accumulated value of the cost of insurance.

Example 5.5 (n-Year Endowment Insurance (...revisited))
Here the apv of the benefit is also given by an annuity:

$$\begin{aligned} {}_tV_{x:\overline{n|}} &= A_{x+t:\overline{n-t|}} - P_{x:\overline{n|}}\,\ddot{a}_{x+t:\overline{n-t|}} \,, \qquad t < n \\ &= P_{x:\overline{n|}}\,\ddot{s}_{x:\overline{t|}} - \frac{A^{1}_{x:\overline{t|}}}{A_{x:\overline{t|}}^{\;\;1}} \,, \qquad t < n \,, \end{aligned}$$

which implies that

$${}_tV_{x:\overline{n|}} - {}_tV_x = (P_{x:\overline{n|}} - P_x)\,\ddot{s}_{x:\overline{t|}} \,.$$

Here the difference in reserves is seen as being the accumulated value of the difference in premiums.

The prospective method is most useful for durations t beyond the premium paying period. For instance:

$$\begin{aligned} {}^h_tV_x &= A_{x+t} \,, \qquad t \geq h \,, \\ {}_tV({}_{n|}\ddot{a}_x) &= \ddot{a}_{x+t} \,, \qquad t \geq n \,. \end{aligned}$$

The retrospective method is most useful during a deferral period. For instance:

$${}_tV({}_{n|}\ddot{a}_x) = P({}_{n|}\ddot{a}_x)\,\ddot{s}_{x:\overline{t|}} \,, \qquad t < n \,.$$

5.4 Additional Formulas

(a) In terms of annuity apv's:

$${}_tV_x = \underbrace{A_{x+t}}_{1-d\,\ddot{a}_{x+t}} - \underbrace{P_x}_{\frac{1}{\ddot{a}_x}-d}\,\ddot{a}_{x+t} = 1 - \frac{\ddot{a}_{x+t}}{\ddot{a}_x} = \frac{\ddot{a}_x - \ddot{a}_{x+t}}{\ddot{a}_x} \,.$$

(b) In terms of insurance NSP's:

$$ {}_tV_x = \frac{\overbrace{\ddot{a}_x}^{\frac{1-A_x}{d}} - \overbrace{\ddot{a}_{x+t}}^{\frac{1-A_{x+t}}{d}}}{\ddot{a}_x} = \frac{A_{x+t} - A_x}{1 - A_x} \; . $$

(c) In terms of discounted premium differences:

$$ \begin{aligned} {}_tV_x &= A_{x+t} - P_x\,\ddot{a}_{x+t} = A_{x+t}\left[1 - \frac{P_x\,\ddot{a}_{x+t}}{A_{x+t}}\right] \\ &= A_{x+t}\left[1 - \frac{P_x}{P_{x+t}}\right] = (P_{x+t} - P_x)\,\ddot{a}_{x+t} \; . \end{aligned} $$

(d) In terms of net premiums:

$$ {}_tV_x = (P_{x+t} - P_x)\,\ddot{a}_{x+t} = \frac{(P_{x+t} - P_x)}{P_{x+t} + d} \; . $$

5.5 Formulas Connecting Successive Reserves

${}_tV_x$ is the terminal reserve at the end of the t-th year. The initial reserve at t (the beginning of the $t+1$-th year) is:

$$ \begin{aligned} {}_tV_x + P_x &= A_{x+t} - P_x\,\ddot{a}_{x+t} + P_x \\ &= A_{x+t} - P_x\,a_{x+t} \\ &= v\,q_{x+t} + v\,p_{x+t}\,A_{x+t+1} - P_x\,v\,p_{x+t}\,\ddot{a}_{x+t+1} \\ &= v\,q_{x+t} + v\,p_{x+t}\,[A_{x+t+1} - P_x\,\ddot{a}_{x+t+1}] \\ &= v\,q_{x+t} + v\,p_{x+t}\;{}_{t+1}V_x \; . \end{aligned} $$

The interpretation is that the initial reserve, accumulated to the end of the year $[{}_tV_x + P_x]\,(1+i)\,l_{x+t}$ must provide a \$1 for each death, $1\,d_{x+t}$, + a new terminal reserve for all survivors, $l_{x+t+1}\;{}_{t+1}V_x$.

Another way to interpret it is

$$ \begin{aligned} {}_tV_x + P_x &= v\,q_{x+t} + v\,(1 - q_{x+t})\;{}_{t+1}V_x \\ &= v\;{}_{t+1}V_x + v\,q_{x+t}\,(1 - {}_{t+1}V_x) \; . \end{aligned} $$

Here $1 - {}_{t+1}V_x$ = the *net amount at risk* for the $(t+1)$-th year.
$q_{x+t}\,(1 - {}_{t+1}V_x)$ = the cost of insurance based upon the net amount at risk.

Example 5.6 (1) For an n-year term endowment:

$$({}_tV_{x:\overline{n}|} + P_{x:\overline{n}|})(1+i) = q_{x+t} + p_{x+t}\,{}_{t+1}V_{x:\overline{n}|}\ .$$

(2) For a whole life insurance:

$$P_x = v\,q_{x+t}\,(1 - {}_{t+1}V_x) + (v\,{}_{t+1}V_x - {}_tV_x)\ .$$