(6.119) & (6.120) become

$$\frac{dA_1}{dz} = -i\kappa A_2(z)\, e^{i\Delta\beta z} \tag{6.124}$$

$$\frac{dA_2}{dz} = -i\kappa^* A_1(z)\, e^{-i\Delta\beta z} \tag{6.125}$$

where, from (6.122), we have set

$$\kappa = C_{12}^{(m)} \tag{6.126}$$

NB. From conservation of energy:

$$\frac{d}{dz}\left\{|A_1(z)|^2 + |A_2(z)|^2\right\} = 0 \tag{6.127}$$

* For $\Delta\beta = 0$:

In (6.121), phase matching occurs because (6.116) is satisfied ...

$$\Delta\beta = \beta_1 - \beta_2 - m\,2\pi/\Lambda = 0 \tag{6.128}$$

... for some integer m.

(6.128) into (6.124) & (6.125) yields

$$\frac{dA_1}{dz} = -i\kappa A_2 \tag{6.129}$$

$$\frac{dA_2}{dz} = -i\kappa^* A_1 \tag{6.130}$$

(6.129) into (6.130) yields

$$\frac{d^2A_1}{dz^2} + |\kappa|^2 A_1 = 0 \tag{6.131}$$

... which has solutions of the form

$$A_1(z) = A_1(0)\cos(|\kappa|z) - \frac{i\kappa}{|\kappa|} A_2(0)\sin(|\kappa|z) \tag{6.132}$$

initial mode amplitudes

(6.132) into (6.129) yields

$$A_2(z) = -\frac{i\kappa^*}{|\kappa|} A_1(0)\sin(|\kappa|z) + A_2(0)\cos(|\kappa|z) \tag{6.133}$$

For a single wave incident at $z = 0$, (6.132) and (6.133) become ...

$$A_1(z) = A_1(0)\cos(|\kappa| z) \tag{6.134}$$

$$A_2(z) = \frac{-i\kappa^*}{|\kappa|} A_1(0)\sin(|\kappa| z) \tag{6.135}$$

(6.134) & (6.135) into (6.127) demonstrates...

$$\frac{d}{dz}\left\{ |A_1(0)|^2 \cos^2(|\kappa| z) + |A_1(0)|^2 \sin^2(|\kappa| z) \right\}$$
$$= \frac{d}{dz}\left\{ |A_1(0)|^2 \right\} = 0$$

... so power is conserved.

For arbitrary L, we define

$$\eta = \left| \frac{A_2(L)}{A_1(0)} \right|^2 = \sin^2(|\kappa| L) \tag{6.136}$$

coupling efficiency

NB. Power transfers back and forth between the two modes, complete transfer occurring for

$$|\kappa| L = \frac{\pi}{2}, \frac{3\pi}{2}, \frac{5\pi}{2}, \ldots \tag{6.137}$$

ie. with a period of π

* For $\Delta\beta \neq 0$:

↳ (6.116) is not satisfied ...

⇒ alter our previous procedure for solving coupled wave equations (6.124) & (6.125).

Rewrite (6.124) as

$$e^{-i\Delta\beta z} \frac{dA_1}{dz} = -i\kappa A_2(z) \quad (6.138)$$

↳ Differentiating (6.138) yields

$$-i\Delta\beta e^{-i\Delta\beta z} \frac{dA_1}{dz} + e^{-i\Delta\beta z} \frac{d^2A_1}{dz^2} = -i\kappa \frac{dA_2}{dz} \quad (6.139)$$

(6.125) into (6.139) yields

$$-i\Delta\beta e^{-i\Delta\beta z} \frac{dA_1}{dz} + e^{-i\Delta\beta z} \frac{d^2A_1}{dz^2}$$

$$= -i\kappa \left\{ -i\kappa^* A_1(z) e^{-i\Delta\beta z} \right\}$$

$$\Rightarrow \frac{d^2A_1}{dz^2} - i\Delta\beta \frac{dA_1}{dz} + |\kappa|^2 A_1(z) = 0 \quad (6.140)$$

The solutions to (6.140), and hence (6.138), are found to be

$$A_1(z) = e^{+i(\Delta\beta/2)z}\left\{\left[\cos(sz) + \frac{i\Delta\beta}{2s}\sin(sz)\right]A_1(0) - \frac{i\kappa}{s}A_2(0)\sin(sz)\right\} \tag{6.141}$$

$$A_2(z) = e^{-i(\Delta\beta/2)z}\left\{-\frac{i\kappa^*}{s}A_1(0)\sin(sz) + \left[\cos(sz) + \frac{i\Delta\beta}{2s}\sin(sz)\right]A_2(0)\right\} \tag{6.142}$$

where: $$s^2 = |\kappa|^2 + (\tfrac{1}{2}\Delta\beta)^2 \tag{6.143}$$

When only $A_1(0) \neq 0$ (single incident mode), (6.141) & (6.142) reduce to

$$A_1(z) = A_1(0)\, e^{+i(\Delta\beta/2)z}\left\{\cos(sz) - \frac{i\Delta\beta}{2s}\sin(sz)\right\} \tag{6.144}$$

$$A_2(z) = A_1(0)\, e^{-i(\Delta\beta/2)z}\left\{-\frac{i\kappa^*}{s}\sin(sz)\right\} \tag{6.145}$$

again, power flow along z is conserved.

see (6.127)

The coupling efficiency, defined in (6.136), can be determined using (6.145) as

$$\eta = \left|\frac{A_2(L)}{A_1(0)}\right|^2 = \frac{|\kappa|^2}{s^2}\sin^2(sL)$$

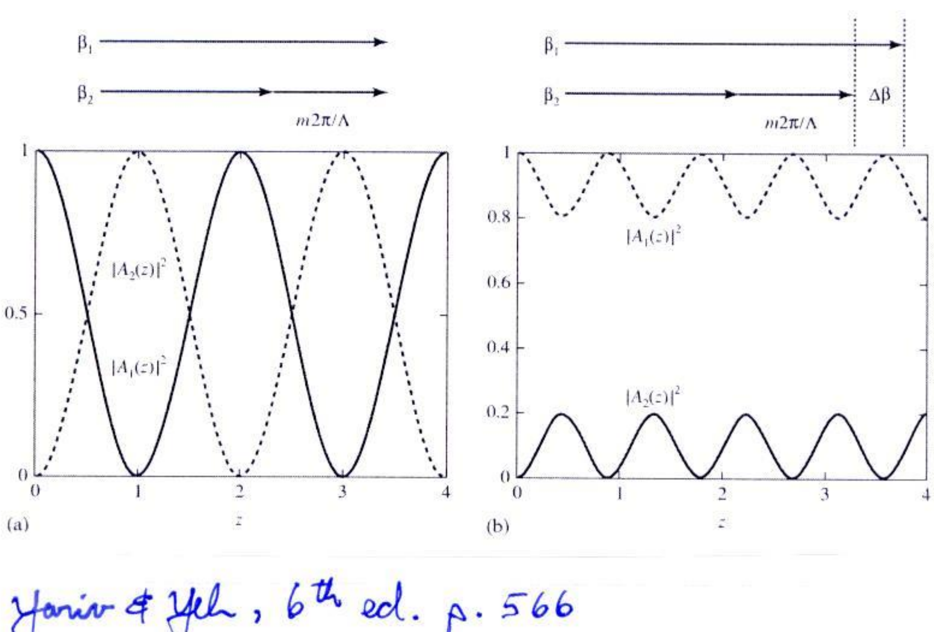
$$= \frac{|\kappa|^2}{|\kappa|^2 + (\frac{1}{2}\Delta\beta)^2}\sin^2\left\{|\kappa| L\sqrt{1+(\tfrac{1}{2}\Delta\beta/|\kappa|)^2}\right\} \quad (6.146)$$

... using (6.143)

(6.146) $\Rightarrow \eta < 1$ for all $\Delta\beta \neq 0$

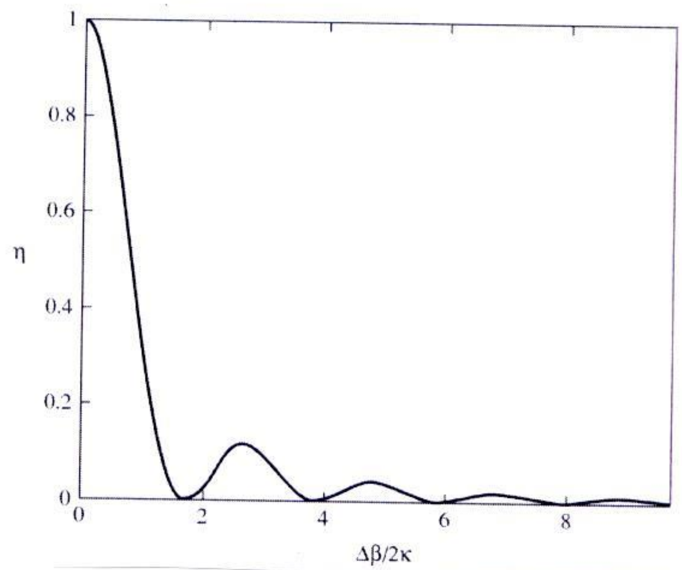
$\hookrightarrow$ incomplete transfer of power due to phase mismatch

Mode powers $|A_1(z)|^2$ & $|A_2(z)|^2$ are plotted below for the phase match & mismatch cases...



Yariv & Yeh, 6th ed. p. 566

Below, we show the coupling efficiency vs phase mismatch, for $|\kappa| L = \pi/2$



Yariv & Yeh, 6th ed., p. 567

Contradirectional Coupling ($\beta_1 \beta_2 < 0$)

Coupled modes propagate in opposite directions

$\Rightarrow \beta_1 > 0$ & $\beta_2 < 0$

$\hookrightarrow$ (6.119) & (6.120) become

$$\frac{dA_1}{dz} = -i\kappa A_2(z)\, e^{i\Delta\beta z} \tag{6.147}$$

$$\frac{dA_2}{dz} = +i\kappa^* A_1(z)\, e^{-i\Delta\beta z} \tag{6.148}$$

NB. Conservation of energy now requires

$$\frac{d}{dz}\left\{ |A_1(z)|^2 - |A_2(z)|^2 \right\} = 0 \qquad (6.149)$$

* For $\Delta\beta = 0$:

(6.147) & (6.148) become

$$\frac{dA_1}{dz} = -i\kappa A_2 \qquad (6.150)$$

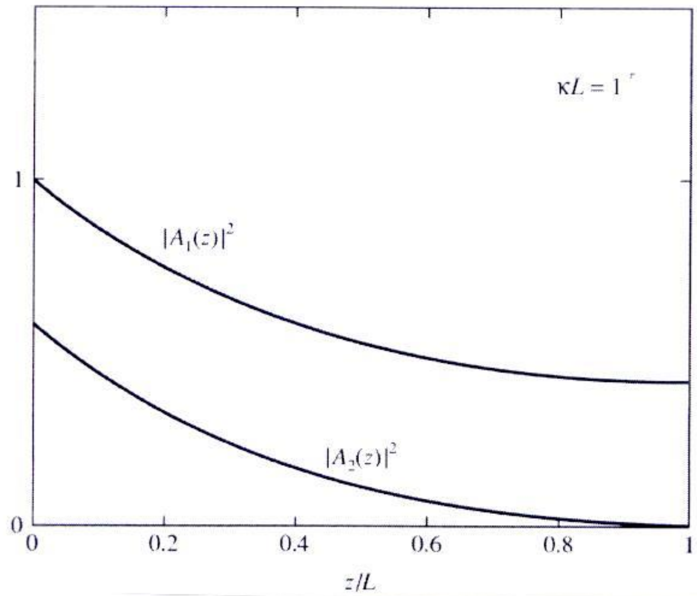
$$\frac{dA_2}{dz} = +i\kappa^* A_1 \qquad (6.151)$$

These coupled mode equations, for $A_2(L) = 0$, have solutions

$$A_1(z) = A_1(0) \cosh[|\kappa|(L-z)] \qquad (6.152)$$

$$A_2(z) = -\frac{iA_1(0)\,\kappa^* \sinh[|\kappa|(L-z)]}{|\kappa| \cosh[|\kappa| L]} \qquad (6.153)$$

The mode powers $|A_1(z)|^2$ & $|A_2(z)|^2$ are shown below.



Yariv & Yeh, 6th ed, p. 569

note the exponential decay of the incident mode and the exponential growth of the back-scattered mode.

In like manner to (6.136), the coupling coefficient is defined as

$$\eta = \left| \frac{A_2(0)}{A_1(0)} \right|^2 \tag{6.154}$$

(6.153) into (6.154) yields

$$\eta = \tanh^2(|\kappa| L) \tag{6.155}$$

$\rightarrow 1$ as $|\kappa| L \rightarrow \infty$

* **For $\Delta\beta \neq 0$:**

By an analysis similar to the previous one, again with $A_2(L) = 0$, the solutions to the coupled mode equations (594) & (595) are

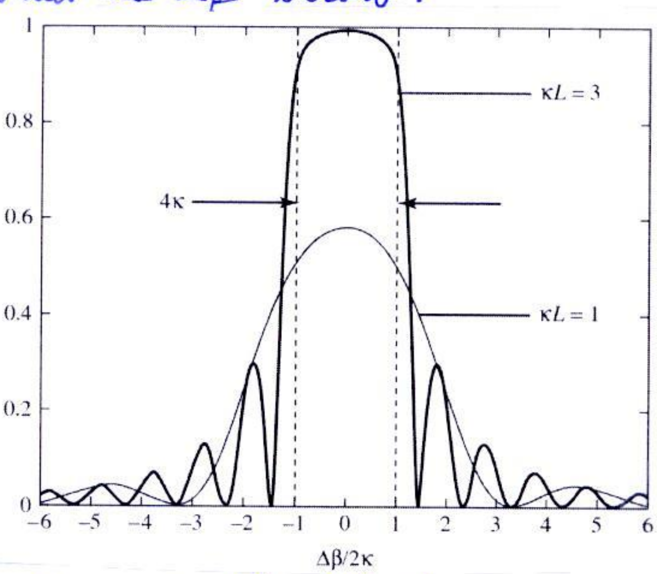
(6.156) $$A_1(z) = A_1(0)\, e^{i\frac{1}{2}\Delta\beta z}\left\{\frac{s\cosh[s(L-z)] + i\frac{1}{2}\Delta\beta\sinh[s(L-z)]}{s\cosh(sL) + i\frac{1}{2}\Delta\beta\sinh(sL)}\right\}$$

(6.157) $$A_2(z) = A_1(0)\, e^{-i\frac{1}{2}\Delta\beta z}\left\{\frac{-i\kappa^*\sinh[s(L-z)]}{s\cosh(sL) + i\frac{1}{2}\Delta\beta\sinh(sL)}\right\}$$

The coupling coefficient, (6.157) into (6.154) at $z = 0$, is

$$\eta = \frac{|\kappa|^2\sinh^2(sL)}{s^2\cosh^2(sL) + (\frac{1}{2}\Delta\beta)^2\sinh^2(sL)} \qquad (6.158)$$

This is plotted vs $\Delta\beta$ below.



Yariv & Yeh, 6th ed, p. 570

NB. Complete power transfer, $\Delta\beta \to 0$ & $L \to \infty$

NB. Unlike codirectional coupling, there is NO periodic exchange of power between the two modes

↳ spectral range of central peak:

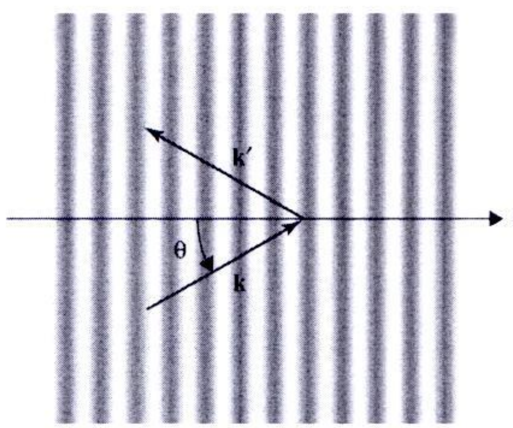
$$-2|\kappa| < \Delta\beta < +2|\kappa| \tag{6.159}$$

Furthermore, we note by comparison of (6.83) & (6.154) that

$$\eta = |r|^2 \tag{6.160}$$

↳ ... the reflectance spectrum

Consider an index grating:



Yariv & Yeh, 6th ed., p. 571

Let the index vary weakly as

$$n(z) = n_0 + n_1 \cos(Kz) \tag{6.161}$$

average index → n_0; index modulation → n_1; $K = \frac{2\pi}{\Lambda}$, not κ!

In terms of the permittivity, its modulation is therefore

$$\Delta\varepsilon(z) = \varepsilon_0 n^2(z) - \varepsilon_0 n_0^2$$

$$\approx 2\varepsilon_0 n_0 n_1 \cos(Kz) \tag{6.162}$$

The Fourier coefficient ε_1 is therefore

$$\varepsilon_1 \approx 2\varepsilon_0 n_0 n_1 \tag{6.163}$$

Since: $\beta = k\cos\theta$ (6.164)

(6.164) into (6.121) yields

$$\Delta\beta = 2k\cos\theta - \frac{2\pi}{\Lambda} \tag{6.165}$$

Let ω_0 be the frequency at which the phase matching condition is satisfied ...

$$\Rightarrow \quad \Delta\beta(\omega_0) = 0 \tag{6.166}$$

Using $\omega = ck/n_0$, (6.165) & (6.166) yield

$$\frac{2\pi}{\Lambda} = 2\frac{n_0\,\omega_0}{c}\cos\theta \tag{6.167}$$

(6.167) into (6.165) yields, for arbitrary ω

$$\Delta\beta = \frac{2n_0}{c}(\omega - \omega_0)\cos\theta \tag{6.168}$$

The half-width of the main peak, as per (6.159), is

$$\Delta\beta_{edge} = 2|\varkappa| \tag{6.169}$$

at normal incidence, using (6.123) & (6.163):

$$\begin{aligned}\varkappa &= \frac{\omega_0^2\mu}{2k}\varepsilon_1 \\ &\approx \omega_0 (n_0\omega/k)\,\varepsilon_0\mu\, n_1 \\ &= \omega_0 n_1/c \end{aligned} \tag{6.170}$$

For $\frac{1}{2}\Delta\omega = \omega_{edge} - \omega_o$, (6.168) & (6.170) into (6.169) yields

$$\frac{2n_o}{c}\left(\frac{1}{2}\Delta\omega\right) = \frac{2\omega_o n_1}{c}$$

$$\Rightarrow \frac{\Delta\omega}{\omega_o} \approx 2\left(\frac{n_1}{n_o}\right)$$

NB. This is consistent with (6.82) in the low contrast limit.