For the special case of $k_y = 0$ ...

normal incidence

...(6.26) & (6.27) yield:

$$k_{1z} = n_1 \omega / c \equiv k_1 \tag{6.64}$$

$$k_{2z} = n_2 \omega / c \equiv k_2 \tag{6.65}$$

$\Rightarrow$ the difference between TE & TM vanishes

$\hookrightarrow$ (6.62) reduces to

$$\cos(K\Lambda) = \cos(k_1 a)\cos(k_2 b) - \frac{1}{2}\left(\frac{n_2}{n_1} + \frac{n_1}{n_2}\right)\sin(k_1 a)\sin(k_2 b) \tag{6.66}$$

Inspection of (6.66) reveals that there will exist ω for which

$$|\cos(K\Lambda)| > 1$$

$\Rightarrow$ photonic band gaps

$\hookrightarrow$ $K\Lambda$ must be interpreted as a complex value

These gaps demark the frequency intervals denoted by

$$\mathrm{Re}\,[K\Lambda] = m\pi \,, \quad m = 1, 2, \ldots \qquad (6.67)$$

Hence, for the 1ST band gap, we may write

$$K\Lambda = \pi \pm i\xi \qquad \text{“Xi”} \qquad (6.68)$$

Let us consider a particular structure, as illustration:

$\rightarrow$ Quarter-Wave Stack (QWS)

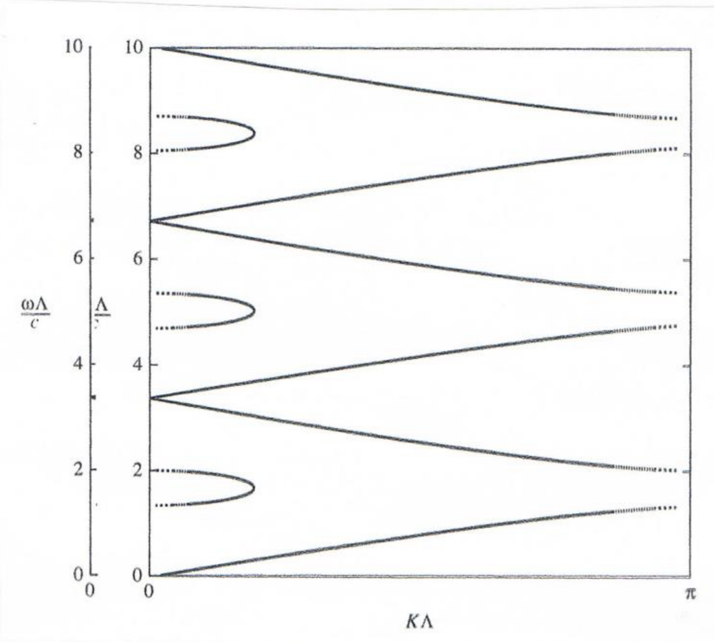
$$a = \frac{\lambda_o}{4n_1} \,, \quad b = \frac{\lambda_o}{4n_2} \qquad (6.69)$$

where: $k_1 a = k_2 b = \pi/2$ (6.70)

for: $\omega_o = \dfrac{2\pi c}{\lambda_o}$ (6.71)

PBG centre

This illustration is plotted below, using dimensionless quantities, and $n_1 = 1.45$, $n_2 = 2.65$, $a = 259$ nm, $b = 142$ nm, $\lambda_o = 1500$ nm



Yariv & Yeh, 6th ed, p. 551

at ω_0, (6.66) becomes

$$\cos(K\Lambda) = -\frac{1}{2}\left(\frac{n_1}{n_2} + \frac{n_2}{n_1}\right) < -1 \qquad (6.72)$$

maximizes ξ

Using (6.68), we can express LHS (6.72) as...

$$\cos(K\Lambda) = \frac{1}{2}\left(e^{iK\Lambda} + e^{-iK\Lambda}\right)$$

$$= \frac{1}{2}\left(e^{-i\pi} e^{\pm\xi} + e^{+i\pi} e^{\mp\xi}\right)$$

$$= -\frac{1}{2}\left(e^{\pm\xi} + e^{\mp\xi}\right) \qquad (6.73)$$

$$= -\cosh\xi$$

equating (6.72) & (6.73) reveals (assuming $n_2 > n_1$)

$$e^{\zeta} = n_2/n_1 \Rightarrow \zeta = \ln(n_2/n_1) \qquad (6.74)$$

For $\Delta n = n_2 - n_1 \approx 0$, using a Taylor series expansion for $\ln x \approx (x-1) - \frac{1}{2}(x-1)^2 + \cdots$

$$\zeta \approx \Delta n / n_1 \qquad (6.75)$$

$\Rightarrow$ evanescent decay rate depends on the relative refractive index contrast

NB. $\zeta = \zeta(\omega)$ – a maximum at PBG centre

this vanishes at the band edge

$\Rightarrow$ can determine $\Delta\omega$

size of the band gap

"Zeta"

Define: $\varsigma = (\omega - \omega_0) n_1 a / c$

$$= (\omega - \omega_0) n_2 b / c \qquad (6.76)$$

frequency deviation (dimensionless)

(6.70) into (6.76) yields, using (6.64) and (6.65)...

$$k_1 a = k_2 b = \zeta + \pi/2 \tag{6.77}$$

(6.68) & (6.77) into (6.66), using (6.73), yields...

$$\cosh \xi = \frac{1}{2}\left(\frac{n_2}{n_1} + \frac{n_1}{n_2}\right)\cos^2\zeta - \sin^2\zeta \tag{6.78}$$

at the band edges, $\xi = 0$, and (6.78) may be solved for ζ as

$$\zeta_{edge} = \pm \sin^{-1}\left(\frac{n_2 - n_1}{n_2 + n_1}\right) \tag{6.79}$$

Using (6.76):

$$\begin{aligned} \zeta_{edge} &= \tfrac{1}{2}\Delta\omega\,(n_1 a/c) \\ &= \tfrac{1}{2}\Delta\omega\,(n_2 b/c) \end{aligned} \tag{6.80}$$

Equating (6.79) & (6.80) yields

$$\frac{\Delta\omega}{2}\left(\frac{n_1 a}{c}\right) = \sin^{-1}\left(\frac{n_2 - n_1}{n_2 + n_1}\right) \tag{6.81}$$

Since (6.64) & (6.70) imply...

$$\frac{n_1 \omega_0 a}{c} = \frac{\pi}{2}$$

... we may rewrite (6.81) as

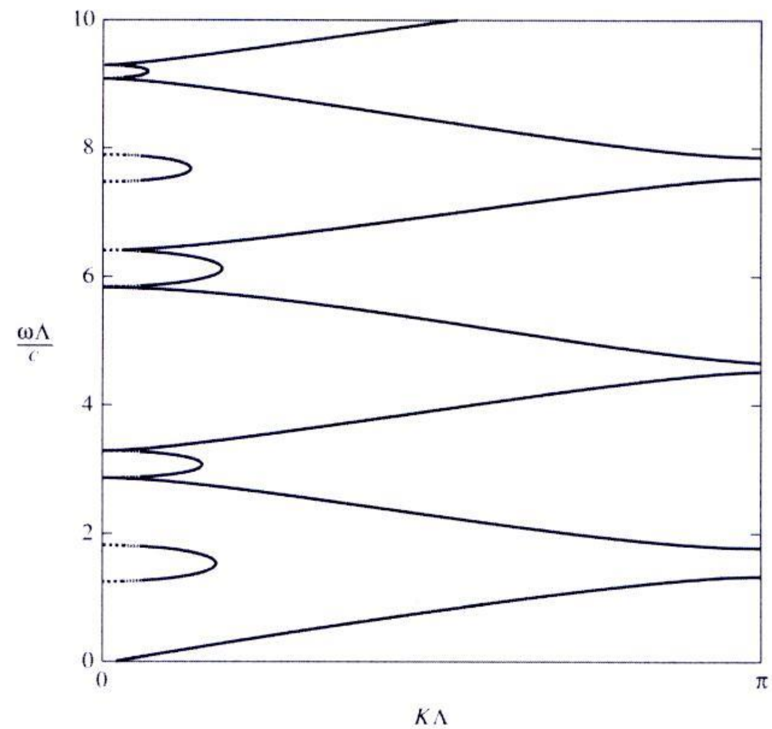
$$\frac{\Delta\omega}{\omega_0} = \frac{4}{\pi}\sin^{-1}\left|\frac{n_2 - n_1}{n_2 + n_1}\right| \qquad (6.82)$$

NB. The QWS is a special stack...

→ the even PBGs vanish
i.e. those for $K\Lambda = 2\pi, 4\pi, \ldots$

→ only odd-order gaps exist

A more general case is shown below, for $n_1 = 1.45$, $n_2 = 2.65$, $a = b = 200$ nm (normal incidence)



Yariv & Yeh, 6th ed, p. 553

Bragg Reflectors

Plane wave   Periodic medium

→ K is generated ←

Bloch wave

→ If ω is in the PBG, K must be evanescent

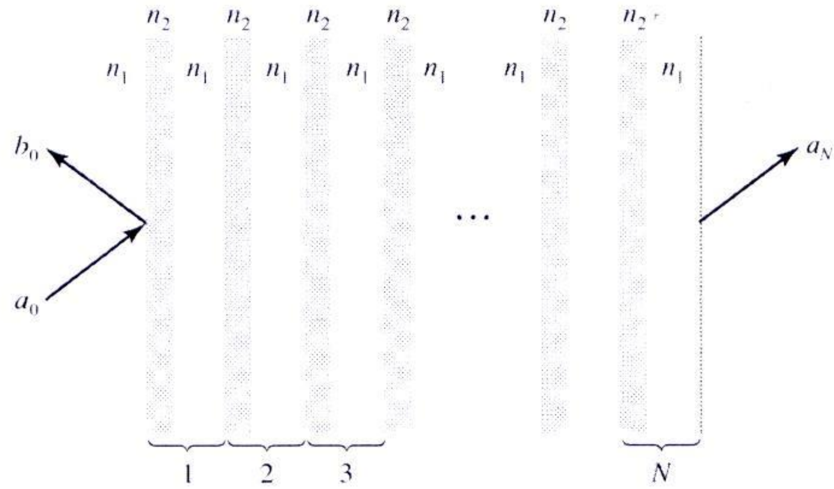
⇒ exponential decay of field amplitude along propagation direction

→ power flow into medium is zero

⇒ incident beam is fully reflected...

... for infinite medium.

Let us consider reflection and transmission for N unit cells embedded in a material with refractive index n_1.



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NB. $b_N = 0$, as there is no wave incident on the exit face.

The reflection coefficient is given by

$$r_N = \left(\frac{b_0}{a_0}\right) \tag{6.83}$$

From (6.45), for $n = N$

$$\begin{bmatrix} a_0 \\ b_0 \end{bmatrix} = \begin{bmatrix} A & B \\ C & D \end{bmatrix}^N \begin{bmatrix} a_N \\ b_N \end{bmatrix} \tag{6.84}$$

This matrix can be rewritten...

Identity:

$$\begin{bmatrix} A & B \\ C & D \end{bmatrix}^N = \begin{bmatrix} AU_{N-1} - U_{N-2} & BU_{N-1} \\ CU_{N-1} & DU_{N-1} - U_{N-2} \end{bmatrix} \tag{6.85}$$

where: $$U_N = \frac{\sin[(N+1)K\Lambda]}{\sin(K\Lambda)} \tag{6.86}$$

(6.85) into (6.84), using $b_N = 0$, allows us to determine a_0 & b_0 in terms of a_N, whence (6.83) may be written as

$$r_N = \frac{C\,U_{N-1}}{A\,U_{N-1} - U_{N-2}} \tag{6.87}$$

The reflectance may be obtained as

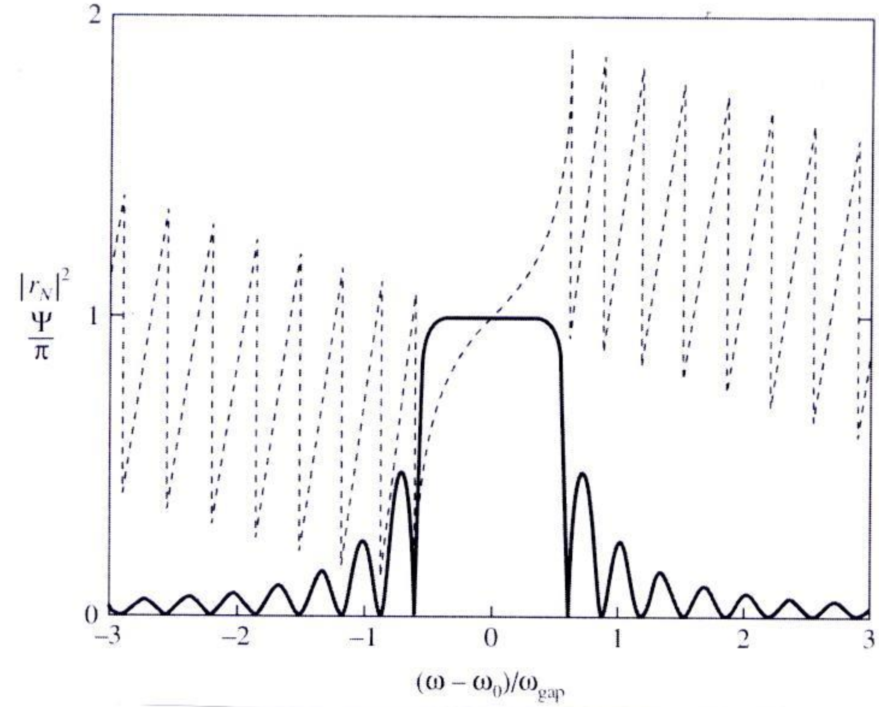
$$|r_N|^2 = \frac{|C|^2}{|C|^2 + \dfrac{\sin^2(K\Lambda)}{\sin^2(NK\Lambda)}} \tag{6.88}$$

From (6.88) we note that at the band edges, where $K\Lambda = m\pi$

$$|r_N|^2_{edge} = \frac{|C|^2}{|C|^2 + (1/N)^2} \tag{6.89}$$

approaches unity for $N \gg 1$

The reflectance spectrum is plotted below, for $N = 10$, $n_1 = 1.45$, $n_2 = 2.25$, $a = 259$, $b = 167$ nm, using the fundamental PBG at $\lambda_0 = 1500$ nm.



Yariv & Yeh, 6th ed., p. 557

NB. In the bandgap, $K\Lambda$ is complex, and (6.88) becomes, generalizing (6.68):

$$|r_N|^2 = \frac{|C|^2}{|C|^2 + \frac{\sinh^2(\xi)}{\sinh^2(N\xi)}} \tag{6.90}$$

$\hookrightarrow$ approaches unity for $N \gg 1$

NB. Since r_N is complex, it may be expressed as

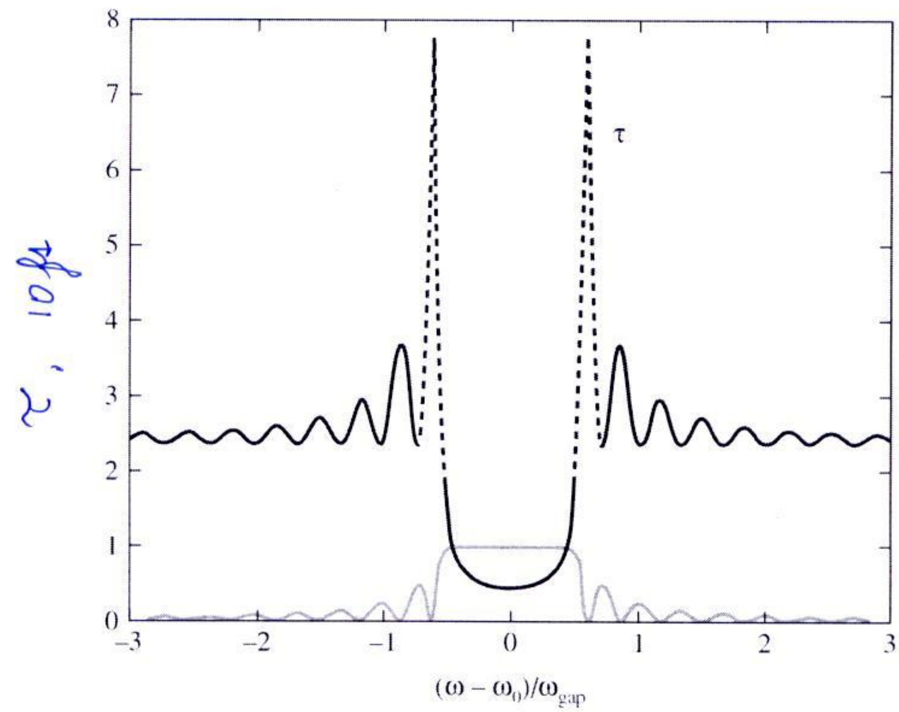
$$r_N = |r_N| \, e^{-i\psi} \tag{6.91}$$

Consistent with (1.155), we may identify

$$\tau = \frac{d\Psi}{d\omega} \tag{6.92}$$

group delay, s

This is plotted below for our previous values.



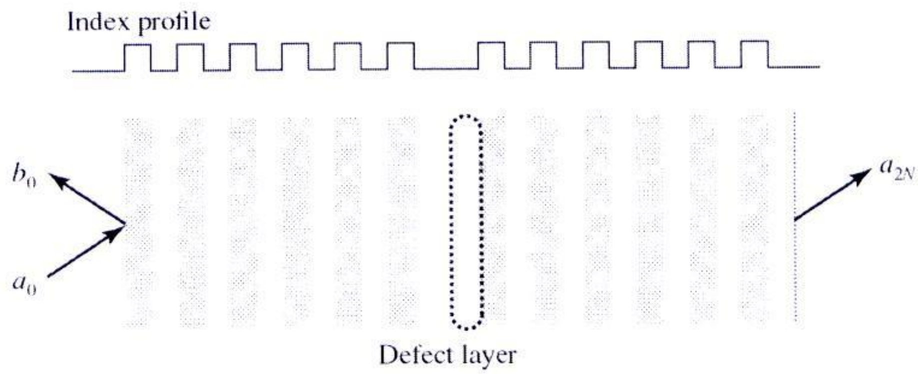
Yariv & Yeh, 6th ed, p. 558

NB. From (1.155): $$v_G = \frac{N(a+b)}{\tau} \approx \frac{c}{n_{av}} \tag{6.93}$$

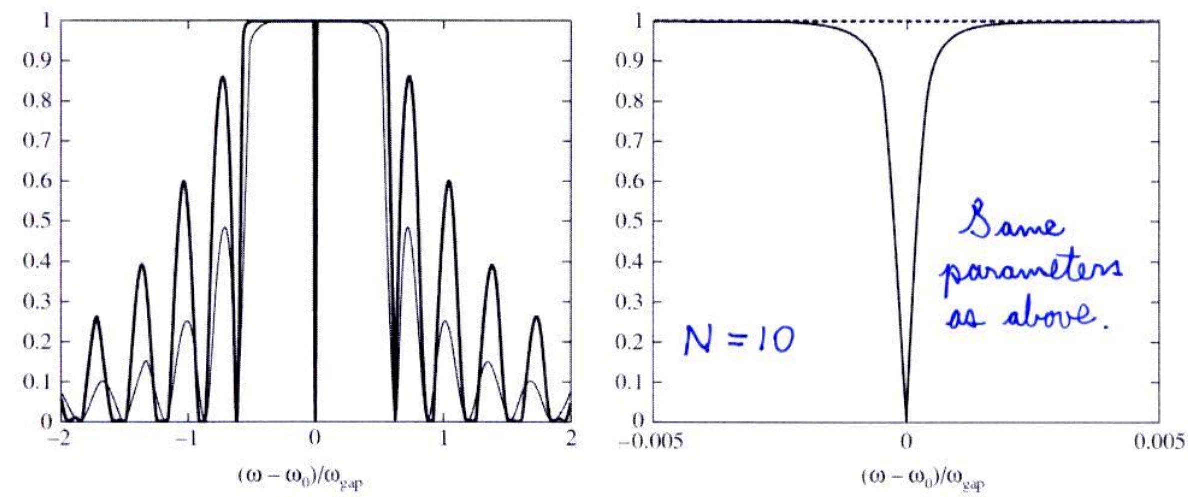
far from the band gap, $n_{av} = \dfrac{n_1 + n_2}{2}$

Transmission Filters

↳ insert a half-wave layer between two QWS ...



Yariv & Yeh, 6th ed, p. 559



The matrix equation then becomes

$$\begin{bmatrix} a_0 \\ b_0 \end{bmatrix} = \begin{bmatrix} A & B \\ C & D \end{bmatrix}^N \begin{bmatrix} e^{ik_1 a} & 0 \\ 0 & e^{-ik_1 a} \end{bmatrix} \begin{bmatrix} A & B \\ C & D \end{bmatrix}^N \begin{bmatrix} a_{2N} \\ 0 \end{bmatrix} \quad (6.94)$$

Coupled Wave Analysis

$\varepsilon(\vec{r}) \leftarrow$ periodic variation

consider as a perturbation that couples the normal modes of the unperturbed system

$$\varepsilon(x,y,z) = \varepsilon_a(x,y) + \Delta\varepsilon(x,y,z) \qquad (6.95)$$

Unperturbed $\longrightarrow \varepsilon_a(x,y)$

$\Delta\varepsilon(x,y,z)$: periodic in z-direction

The known normal modes, from (5.3), are

$$\vec{\mathcal{E}}_m(\vec{r},t) = \vec{E}_m(x,y)\, e^{i(\omega t - \beta_m z)} \qquad (6.96)$$

Wavefunction $\longrightarrow \vec{E}_m(x,y)$

β_m: propagation constant, m^{th} mode

Let's assume the wavefunction satisfies a TE mode equation ...

$$\left\{ \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \omega^2 \mu\, \varepsilon_a(x,y) - \beta_m^2 \right\} \vec{E}_m(x,y) = 0 \qquad (6.97)$$

(... ie. the E_y formulation of (5.63), for $\vec{\nabla}\cdot\vec{E} = 0$)

The excitation of an arbitrary field of frequency ω may be expressed as

$$\vec{\mathcal{E}}(\vec{r},t) = \sum_m A_m \vec{E}_m(x,y)\, e^{i(\omega t - \beta_m z)} \qquad (6.98)$$

constant coefficient

linear combination of normal modes

NB. The orthonormalization condition of (5.60) applies to all $\vec{E}_m$, whence

$$\int_S \vec{E}_n^{\,*}(x,y)\cdot\vec{E}_m(x,y)\, dS = \frac{2\omega\mu}{|\beta_m|}\,\delta_{mn} \qquad (6.99)$$

Suppose we excite an unperturbed mode ...

$$\vec{E}_1(x,y)\, e^{i(\omega t - \beta_1 z)}$$

... which propagates in a periodically perturbed medium

$\Longrightarrow$ induces an additional polarization $\Delta\vec{P}$ ...

From (1.73), (1.74) & (6.95), this is...

$$\Delta\vec{P} = \Delta\varepsilon(x,y,z)\,\vec{E}_1(x,y)\,e^{i(\omega t-\beta_1 z)} \qquad (6.100)$$

acts as a source term ... see (1.79)

$\Rightarrow$ may alter modal power in $\vec{E}_2$

ie. $\vec{E}_1$ & $\vec{E}_2$ may "couple"

exchange energy

now, for media with $\Delta\varepsilon \neq 0$...

$$\vec{E}_m(x,y)\,e^{i(\omega t-\beta_m z)}$$

... no longer eigenmodes

But, if $\Delta\varepsilon$ is small, then perturbation theory can use the known modes to build the new ones...

... express the arbitrary field via (6.98) using $A_m \longrightarrow A_m(z)$:

spatially - dependent expansion coefficients

$$\vec{\mathcal{E}}(\vec{r},t) = \sum_m \underset{\substack{\uparrow \\ \text{addresses the coupling}}}{A_m(z)} \underbrace{\vec{E}_m(x,y)\, e^{i(\omega t - \beta_m z)}}_{\text{basis of unperturbed modes}} \tag{6.101}$$

Considering just the spatial part of (6.101):

$$\vec{E}(\vec{r}) = \sum_m A_m(z)\vec{E}_m(x,y)e^{-i\beta_m z} \tag{6.102}$$

Using (6.95), the Helmholtz equation, (4.1) , is

$$\{\nabla^2 + \omega^2\mu[\varepsilon_a(x,y) + \Delta\varepsilon(x,y,z)]\}\vec{E} = 0 \tag{6.103}$$

(6.102) into (6.103) yields...

$$\{\nabla_t^2 + \frac{d^2}{dz^2} + \omega^2\mu\varepsilon_a + \omega^2\mu\Delta\varepsilon\}$$

$$\cdot\{\sum_m A_m(z)\vec{E}_m(x,y)e^{-i\beta z}\} = 0$$

$$\sum_m A_m(z)\{[\nabla_t^2 + \omega^2\mu\varepsilon_a - \beta_m]\vec{E}_m(x,y)\}e^{-i\beta_m z} \tag{6.104}$$
$$+ \sum_m \left\{\frac{d^2}{dz^2} + \omega^2\mu\Delta\varepsilon + \beta_m\right\}[A_m(z)\vec{E}_m(x,y)e^{-i\beta_m z}] = 0$$

From (6.97), the first summation in (6.104) vanishes, and the second can be written as...

$$\sum_m \left\{ \frac{d^2}{dz^2} A_m(z) - 2i\beta_m \frac{d}{dz} A_m(z) \right\} \vec{E}_m(x,y)\, e^{-i\beta_m z} \tag{6.105}$$

$$= -\omega^2 \mu \sum_n \Delta\varepsilon(x,y,z) A_n(z) \vec{E}_n(x,y)\, e^{-i\beta_n z}$$

We now assume that the perturbation is "small":

$$\frac{d^2}{dz^2} A_m \ll \beta_m \frac{d}{dz} A_m \tag{6.106}$$

Slowly varying amplitude (SVA) approximation

applying (6.106) to (6.105) yields

$$-2i \sum_m \beta_m \left(\frac{d}{dz} A_m(z) \right) \vec{E}_m(x,y)\, e^{-i\beta_m z} \tag{6.107}$$

$$= -\omega^2 \mu \sum_n \Delta\varepsilon(x,y,z) A_n(z) \vec{E}_n(x,y)\, e^{-i\beta_n z}$$

Taking the dot product with $\vec{E}_k^*(x,y)$ and integrating over the transverse cross section...

$$-2i \sum_m \beta_m \left(\frac{d}{dz} A_m(z) \right) e^{-i\beta_m z} \int_S \vec{E}_k^* \cdot \vec{E}_m \, dS \tag{6.108}$$

$$= -\omega^2 \mu \sum_n A_n(z)\, e^{-i\beta_n z} \int_S \vec{E}_k^* \cdot \Delta\varepsilon \vec{E}_n \, dS$$

Using the orthonormalization condition (6.99) into (6.108):

$$\frac{4i\beta_k}{|\beta_k|} e^{-i\beta_k z} \frac{d}{dz} A_k(z) = \omega \sum_m A_m(z) e^{-i\beta_m z} \times \int_S \vec{E}_k^* \cdot \Delta\varepsilon \vec{E}_m \, dS \quad (6.109)$$

Now, expand $\varepsilon(\vec{r})$ as a Fourier series:

$$\varepsilon(\vec{r}) = \sum_m \varepsilon_m(x,y) e^{-im(2\pi/\Lambda)z}$$

$\hookrightarrow m = 0 \Rightarrow \varepsilon_a(x,y)$

... the average (unperturbed) permittivity

$$\Rightarrow \Delta\varepsilon(x,y,z) = \sum_{m\neq 0} \underbrace{\varepsilon_m(x,y)}_{\text{Fourier coefficients}} e^{-im(2\pi/\Lambda)z} \quad (6.110)$$

(6.110) into (6.109) yields...

$$\frac{d}{dz} A_k(z) = -\frac{\omega\beta_k i}{4|\beta_k|} \sum_{m\neq 0} \sum_n \left\{ \int_S \vec{E}_k^* \cdot \varepsilon_m(x,y) \vec{E}_n \, dS \right\} \times A_n(z) e^{i(\beta_k - \beta_n - m2\pi/\Lambda)z} \quad (6.111)$$

Let us identify ...

$$C_{kn}^{(m)} = \frac{\omega}{4} \int_S \vec{E}_k^{\,*}(x,y) \cdot \mathcal{E}_m(x,y)\, \vec{E}_n(x,y)\, dS \qquad (6.112)$$

magnitude of the coupling between k^{th} & n^{th} modes due to the m^{th} Fourier component of the perturbation

... whence (6.111) becomes

$$\frac{d}{dz} A_k(z) = -\frac{i\beta_k}{|\beta_k|} \sum_{m \neq 0} \sum_n C_{kn}^{(m)} A_n(z) \times e^{i(\beta_k - \beta_n - m\,2\pi/\Lambda)z} \qquad (6.113)$$

an infinite set of coupled differential equations ...

... usually only two modes are strongly coupled

Let us "unpack" (6.113) to better draw out its meaning:

The change in the field amplitude of the k^{th} mode:

$\longrightarrow dA_k$

... arising from coupling to the n^{th} mode:

$$\rightarrow A_n(z)$$

... via the m^{th} Fourier component:

$$\rightarrow C_{kn}^{(m)}$$

... across the interval $z \rightarrow z + \Delta z$ is, from (6.113)

$$dA_k = \frac{-i\beta_k}{|\beta_k|} C_{kn}^{(m)} A_n(z) e^{i(\beta_k - \beta_n - m2\pi/\Lambda)z} dz \tag{6.114}$$

... slowly varying over $L \gg \Lambda$

$\Rightarrow$ Integrating (6.114) over L yields

$$\Delta A_k = \frac{-i\beta_k}{|\beta_k|} C_{km}^{(m)} A_n(z) \int_{L \gg \Lambda} e^{i(\beta_k - \beta_n - m2\pi/\Lambda)z} dz \tag{6.115}$$

This integral vanishes
if the exponent in the
exponential is non-zero...

Hence, for notable coupling to occur ...

$$\beta_k - \beta_m - m\,2\pi/\Lambda = 0 \tag{6.116}$$

"Longitudinal phase matching"

Kinematical condition

... and:

$$C_{km}^{(m)} \neq 0 \tag{6.117}$$

Dynamical condition

The Coupled Mode Equations

Let the two coupled modes be designated as

$$\vec{\mathcal{E}}_1(\vec{r},t) = \vec{E}_1(x,y)\, e^{i(\omega t - \beta_1 z)}$$

$$\mathcal{E}_2(\vec{r},t) = \vec{E}_2(x,y)\, e^{i(\omega t - \beta_2 z)} \tag{6.118}$$

neglecting interaction with any modes other than $\mathbf{E}_2$ ($\mathbf{E}_1$)

(6.118) into (6.113) yields

$$\frac{dA_1}{dz} = \frac{-i\beta_1}{|\beta_1|} C_{12}^{(m)} A_2(z)\, e^{i\Delta\beta z} \tag{6.119}$$

$$\frac{dA_2}{dz} = \frac{-i\beta_2}{|\beta_2|} C_{21}^{(-m)} A_1(z)\, e^{-i\Delta\beta z} \tag{6.120}$$

where: $\Delta\beta = \beta_1 - \beta_2 - m\,2\pi/\Lambda$

$$= \beta_1 - \beta_2 - mK \qquad (6.121)$$

grating wave number, cf. (6.5)

NB. It is apparent, from (6.112), that...

$$C_{12}^{(m)} = \left[C_{21}^{(-m)} \right]^* \qquad (6.122)$$

... if $\Delta\varepsilon$ is Hermitian.

If $\Delta\varepsilon = \Delta\varepsilon(z)$, then the normal modes of the unperturbed medium are plane waves, and the Fourier coefficients are constants. Then (6.112) becomes

$$C_{kn}^{(m)} = \frac{\omega^2 \mu}{2\sqrt{|\beta_k \beta_n|}} \, \vec{p}_k^{\,*} \cdot \varepsilon_m \, \vec{p}_n \qquad (6.123)$$

where $\vec{p} = \vec{E}/|\vec{E}|$ denotes the polarization unit vector.

Codirectional Coupling ($\beta_1 \beta_2 > 0$)

Coupled modes propagate in the same direction...

$$\Rightarrow \beta_1 \;\&\; \beta_2 > 0$$