Ch.6 Wave propagation in periodic media

6.1 Periodic media – general 1D description

... a heavily exploited situation ...

→ Diffraction gratings...
Bragg mirrors...
Filters...

& more recently...

- VCSEL
- DFB lasers
- DBR lasers
- FBG
- PhC

Bragg Law & Grating Equation

... recall basics of Bragg scattering

↳ Consider a kinematic description of scattering for a 1-D medium, periodic.

Equidistant infinite planes

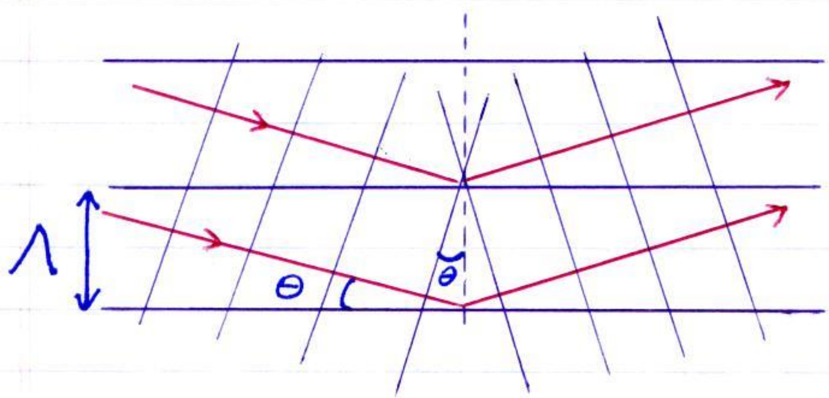
↳ "Thick grating" - transverse plane dimensions >> beam size

... Λ is the period of the index variation

Total scattered wave

↳ linear superposition of all partially reflected plane waves

Constructive interference:



Let n be the average refractive index

$$\Delta OPL = 2\pi m \qquad (6.1)$$

integer

change in optical path length

$$\Delta OPL = k \Delta L$$

$$= \frac{2\pi n}{\lambda} \cdot 2\Lambda \sin\theta \qquad (6.2)$$

Equating (6.1) & (6.2):

$$2\Lambda \sin\theta = \frac{m\lambda}{n} \qquad (6.3)$$

"Bragg law"

for any θ that satisfies (6.3), there is one diffraction order, m.

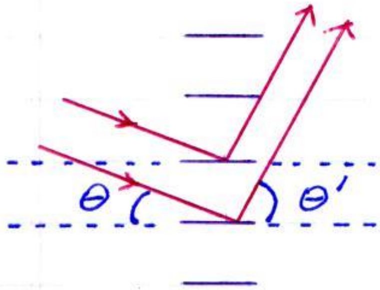
alternatively, (6.3) may be written as

$$2k \sin\theta = m G \qquad (6.4)$$

where: $$G = \frac{2\pi}{\Lambda} \qquad (6.5)$$

grating wave number

"Thin Grating" ← transverse plane dimensions ~ beam size



Specular reflection + diffraction ...

⇒ scattered light directed at

$$\theta' \neq \theta$$

Constructive interference when

$$\Lambda \sin\theta + \Lambda \sin\theta' = \frac{m\lambda}{n} \qquad (6.6)$$

↑ "Grating equation"

↳ for any θ that satisfies (6.6), there are a series of diffraction orders $\theta' \leftrightarrow m$

Let us formally consider such periodic media.

6.2 Bloch's Theorem

Consider a structure whose permittivity is a periodic function...

$$\varepsilon(\vec{r}) = \varepsilon(\vec{r} + \vec{a}) \tag{6.7}$$

an arbitrary lattice vector

We may expand $\varepsilon(\vec{r})$ in a Fourier series

$$\varepsilon(\vec{r}) = \sum_{\vec{G}} \varepsilon_{\vec{G}} \, e^{-i\vec{G}\cdot\vec{r}} \tag{6.8}$$

reciprocal lattice vector

Fourier expansion coefficient

e.g. In 1-D: $\vec{G} = m\left(\frac{2\pi}{\Lambda}\right)\hat{z}$, (6.9)

for $m = 0, \pm 1, \pm 2, \ldots$

We may express the general solution to Maxwell's equations in periodic media using a Fourier integral representation:

$$\vec{E}(\vec{r}) = \int d\vec{k} \, \underbrace{\vec{A}(\vec{k})} \, e^{-i\vec{k}\cdot\vec{r}} \tag{6.10}$$

amplitude of the plane wave component of wavevector $\vec{k}$

As before, the eigenvalue equation is

$$\vec{\nabla} \times \vec{\nabla} \times \vec{E} = \omega^2 \mu \varepsilon \vec{E} \tag{6.11}$$

(6.8) & (6.10) into (6.11) yields

$$\int d\vec{k}\,(-i\vec{k})\times[-i\vec{k}\times\vec{A}(\vec{k})]\,e^{-i\vec{k}\cdot\vec{r}} - \omega^2\mu\sum_{\vec{G}}\int d\vec{k}\,\varepsilon_{\vec{G}}\vec{A}(\vec{k})\,e^{-i(\vec{k}+\vec{G})\cdot\vec{r}} = 0$$

$$\int d\vec{k}\left\{\vec{k}\times[\vec{k}\times\vec{A}(\vec{k})] + \omega^2\mu\sum_{\vec{G}}\varepsilon_{\vec{G}}\vec{A}(\vec{k}-\vec{G})\right\}e^{-i\vec{k}\cdot\vec{r}} = 0 \qquad (6.12)$$

Since (6.12) holds for all $\vec{r}$,

$$\vec{k}\times[\vec{k}\times\vec{A}(\vec{k})] + \omega^2\mu\sum_{\vec{G}}\varepsilon_{\vec{G}}\vec{A}(\vec{k}-\vec{G}) = 0 \qquad (6.13)$$

(6.13) $\Rightarrow$ only those Fourier components $\vec{A}(\vec{k})$ related by the reciprocal lattice vectors $\vec{G}$ constitute the eigenvalue problem

$\hookrightarrow$ In (6.10), we only need those $\vec{k} = \vec{K}$ where $\vec{K} = \vec{K} - \vec{G}$

Bloch wavevector

$\Rightarrow \vec{E}_{\vec{K}}(\vec{r})$ is a normal mode, every one of which has the form, from (6.10):

$$
\begin{aligned}
\vec{E}_{\vec{K}}(\vec{r}) &= \sum_{\vec{G}} A(\vec{K}-\vec{G})\, e^{-i(\vec{K}-\vec{G})\cdot\vec{r}} \\
&= e^{-i\vec{K}\cdot\vec{r}} \sum_{\vec{G}} \vec{A}(\vec{K}-\vec{G})\, e^{i\vec{G}\cdot\vec{r}} \\
&= \vec{u}_{\vec{K}}(\vec{r})\, e^{-i\vec{K}\cdot\vec{r}}
\end{aligned}
\tag{6.14}
$$

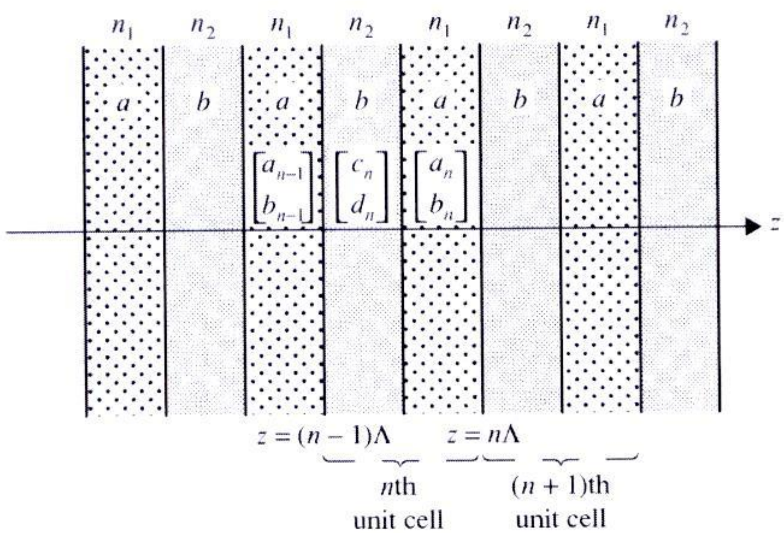
where

$$\vec{u}_{\vec{K}}(\vec{r}) = \sum_{\vec{G}} \vec{A}(\vec{K}-\vec{G})\, e^{i\vec{G}\cdot\vec{r}} \tag{6.15}$$

is a periodic function that satisfies

$$\vec{u}_{\vec{K}}(\vec{r}+\vec{a}) = \vec{u}_{\vec{K}}(\vec{r}) \tag{6.16}$$

6.3 Bloch Waves in Periodic Layered Media



Yariv & Yeh, 6th ed., p. 546

Periodicity is described by

$$n(z) = \begin{cases} n_2 \,, & 0 < z < b \\ n_1 \,, & b < z < \Lambda \end{cases} \tag{6.17}$$

with $$n(z) = n(z+\Lambda) \tag{6.18}$$

where $$\Lambda = a + b \tag{6.19}$$

period

WOLOG, consider propagation in the yz-plane. a general solution to the wave equation will have the form...

$$\vec{\mathcal{E}}(y,z,t) = \vec{E}(z)\, e^{i(\omega t - k_y y)} \tag{6.20}$$

... since there is continuous translational symmetry in the y-direction.

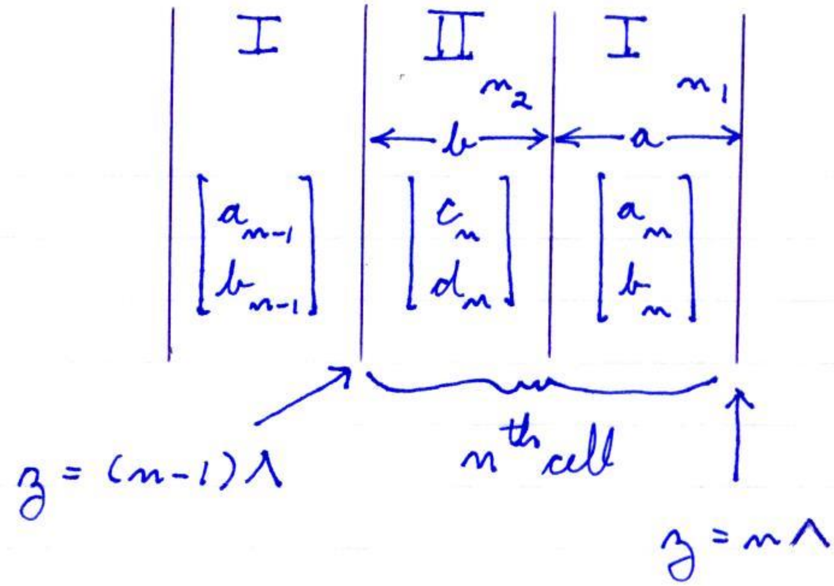
Consider the TE wave (aka: s-wave)

→ $\vec{E} \perp$ to yz-plane, $\vec{E} = \hat{x} E$

plane of propagation

→ both forward and backward propagating components exist...

... in the n^{th} unit cell:



For I: $n\Lambda - a < z < n\Lambda$

$$\Rightarrow 0 < n\Lambda - z < a$$

For II: $(n-1)\Lambda < z < n\Lambda - a$

$$\Rightarrow 0 < n\Lambda - (a+z) < b$$

The electric field within each layer then has the form

$$E_x(z) = \begin{cases} a_n e^{ik_{1z}(n\Lambda - z)} + b_n e^{-ik_{1z}(n\Lambda - z)}, & \text{I} \\ c_n e^{ik_{2z}[n\Lambda - (a+z)]} + d_n e^{-ik_{2z}[n\Lambda - (a+z)]}, & \text{II} \end{cases} \tag{6.21}$$

(6.21) is a solution to

$$\frac{\partial^2 E_x}{\partial z^2} + (k^2 n^2 - k_y^2) E_x = 0 \tag{6.22}$$

...see (5.13)

with: $$H_y = \frac{-1}{i\omega\mu_0} \frac{\partial E_x}{\partial z} \tag{6.23}$$

$$H_z = \frac{-k_y}{\omega\mu_0} E_x \tag{6.24}$$

and: $$E_y = E_z = H_x = 0 \tag{6.25}$$

(6.21) into (6.22) yields

$$k_{1z} = \sqrt{(kn_1)^2 - k_y^2} \tag{6.26}$$

$$k_{2z} = \sqrt{(kn_2)^2 - k_y^2} \tag{6.27}$$

The boundary conditions are the continuity of the tangential components E_x & H_y

→ determine the relation between the forward & backward field components at $z = (n-1)\Lambda$ & $z = n\Lambda - a$

In the $\underline{\underline{3}}$ layers, (6.21) generalizes as

$$E_x(z) = \begin{cases} a_{n-1}\, e^{ik_{1z}[(n-1)\Lambda - z]} + b_{n-1}\, e^{-ik_{1z}[(n-1)\Lambda - z]} \\ c_n\, e^{ik_{2z}[n\Lambda - (a+z)]} + d_n\, e^{-ik_{2z}[n\Lambda - (a+z)]} \\ a_n\, e^{ik_{1z}[n\Lambda - z]} + b_n\, e^{-ik_{1z}[n\Lambda - z]} \end{cases} \tag{6.28}$$

For $z = (n-1)\Lambda$:

$$a_{n-1} + b_{n-1} = c_n e^{ik_{2z}b} + d_n e^{-ik_{2z}b} \qquad (6.29)$$

$$ik_{1z}(a_{n-1} - b_{n-1}) = ik_{2z}(c_n e^{ik_{2z}b} - d_n e^{-ik_{2z}b}) \qquad (6.30)$$

For $z = n\Lambda - a$:

$$c_n + d_n = a_n e^{ik_{1z}a} + b_n e^{-ik_{1z}a} \qquad (6.31)$$

$$ik_{2z}(c_n - d_n) = ik_{1z}(a_n e^{ik_{1z}a} - b_n e^{-ik_{1z}a}) \qquad (6.32)$$

(6.29) & (6.30) may be written as

$$\begin{bmatrix} 1 & 1 \\ ik_{1z} & -ik_{1z} \end{bmatrix} \begin{bmatrix} a_{n-1} \\ b_{n-1} \end{bmatrix} = \begin{bmatrix} e^{ik_{2z}b} & e^{-ik_{2z}b} \\ ik_{2z}e^{ik_{2z}b} & -ik_{2z}e^{-ik_{2z}b} \end{bmatrix} \begin{bmatrix} c_n \\ d_n \end{bmatrix} \qquad (6.33)$$

(6.31) & (6.32), likewise

$$\begin{bmatrix} 1 & 1 \\ ik_{2z} & -ik_{2z} \end{bmatrix} \begin{bmatrix} c_n \\ d_n \end{bmatrix} = \begin{bmatrix} e^{ik_{1z}a} & e^{-ik_{1z}a} \\ ik_{1z}e^{ik_{1z}a} & -ik_{1z}e^{-ik_{1z}a} \end{bmatrix} \begin{bmatrix} a_n \\ b_n \end{bmatrix} \qquad (6.34)$$

By matrix inversion, we may solve (6.34) for $[c_n, d_n]$. Substituting that into (6.33) will yield a matrix equation of the form

$$\begin{bmatrix} a_{n-1} \\ b_{n-1} \end{bmatrix} = \begin{bmatrix} A & B \\ C & D \end{bmatrix} \begin{bmatrix} a_n \\ b_n \end{bmatrix} \tag{6.35}$$

where

$$A = e^{ik_{1z}a}\left\{\cos(k_{2z}b) + \frac{i}{2}\left[\frac{k_{2z}}{k_{1z}} + \frac{k_{1z}}{k_{2z}}\right]\sin(k_{2z}b)\right\} \tag{6.36}$$

$$B = e^{-ik_{1z}a}\left\{\frac{i}{2}\left[\frac{k_{2z}}{k_{1z}} - \frac{k_{1z}}{k_{2z}}\right]\sin(k_{2z}b)\right\} \tag{6.37}$$

$$C = e^{ik_{1z}a}\left\{\frac{-i}{2}\left[\frac{k_{2z}}{k_{1z}} - \frac{k_{1z}}{k_{2z}}\right]\sin(k_{2z}b)\right\} \tag{6.38}$$

$$D = e^{-ik_{1z}a}\left\{\cos(k_{2z}b) - \frac{i}{2}\left[\frac{k_{2z}}{k_{1z}} + \frac{k_{1z}}{k_{2z}}\right]\sin(k_{2z}b)\right\} \tag{6.39}$$

$\begin{bmatrix} A & B \\ C & D \end{bmatrix} \rightarrow$ a <u>unit cell</u> translation matrix

$\Rightarrow$ unimodular

$$\text{ie. } \begin{vmatrix} A & B \\ C & D \end{vmatrix} = AD - BC = 1 \tag{6.40}$$

(6.35) also holds for TM waves, but with different matrix elements, as below:

$$A = e^{ik_{1z}a}\left\{\cos(k_{2z}b) + \frac{i}{2}\left[\frac{n_2^2 k_{1z}}{n_1^2 k_{2z}} + \frac{n_1^2 k_{2z}}{n_2^2 k_{1z}}\right]\sin(k_{2z}b)\right\} \tag{6.41}$$

$$B = e^{-ik_{1z}a}\left\{\frac{i}{2}\left(\frac{n_2^2 k_{1z}}{n_1^2 k_{2z}} - \frac{n_1^2 k_{2z}}{n_2^2 k_{1z}}\right)\sin(k_{2z}b)\right\} \tag{6.42}$$

$$C = e^{ik_{1z}a}\left\{-\frac{i}{2}\left(\frac{n_2^2 k_{1z}}{n_1^2 k_{2z}} - \frac{n_1^2 k_{2z}}{n_2^2 k_{1z}}\right)\sin(k_{2z}b)\right\} \tag{6.43}$$

$$D = e^{-ik_{1z}a}\left\{\cos(k_{2z}b) - \frac{i}{2}\left[\frac{n_2^2 k_{1z}}{n_1^2 k_{2z}} + \frac{n_1^2 k_{2z}}{n_2^2 k_{1z}}\right]\sin(k_{2z}b)\right\} \tag{6.44}$$

(6.35) may be successively applied to traverse the multilayer, one bilayer (unit cell) per "step", yielding

$$\begin{bmatrix} a_0 \\ b_0 \end{bmatrix} = \begin{bmatrix} A & B \\ C & D \end{bmatrix}^n \begin{bmatrix} a_n \\ b_n \end{bmatrix} \tag{6.45}$$

Using matrix inversion & (6.40), (6.35) may also be written as

$$\begin{bmatrix} a_n \\ b_n \end{bmatrix} = \begin{bmatrix} D & -B \\ -C & A \end{bmatrix}^n \begin{bmatrix} a_0 \\ b_0 \end{bmatrix} \tag{6.46}$$

6.4 Bloch Waves & Photonic Band Structures

→ Optical propagation in periodic media

→ Electron propagation in crystalline solids

⇒ use same physical concepts

↳ condensed matter physics

Bloch waves, Brillouin zone, energy bands...

Expressing $\vec{E}(z)$ in (6.20) by Bloch's theorem, (6.14):

$$\vec{E}(z) = \vec{u}_K(z)\, e^{-iKz} \tag{6.47}$$

... and, (6.16):

$$\vec{u}_K(z) = \vec{u}_K(z+\Lambda) \tag{6.48}$$

↳ How to find K & $\vec{u}_K(z)$ in terms of ω & k_y?

Considering (6.28) in light of (6.48), the periodic condition required by (6.47) may be expressed as

$$\begin{bmatrix} a_n \\ b_n \end{bmatrix} = e^{-iK\Lambda} \begin{bmatrix} a_{n-1} \\ b_{n-1} \end{bmatrix} \tag{6.49}$$

Equating (6.49) & (6.35) for the $(n-1)$ column vector:

$$\begin{bmatrix} A & B \\ C & D \end{bmatrix} \begin{bmatrix} a_n \\ b_n \end{bmatrix} = e^{+iK\Lambda} \begin{bmatrix} a_n \\ b_n \end{bmatrix} \qquad (6.50)$$

$\hookrightarrow$ This is an eigenvalue problem.

Hence:

$$\det \begin{vmatrix} A - e^{iKA} & B \\ C & D - e^{iKA} \end{vmatrix} = 0 \qquad (6.51)$$

$$\Rightarrow (e^{iK\Lambda})^2 - (A+D)\, e^{iK\Lambda} + 1 = 0 \qquad (6.52)$$

(6.52) has the solution:

$$e^{iK\Lambda} = \tfrac{1}{2}(A+D) \pm \sqrt{\tfrac{1}{4}(A+D)^2 - 1} \qquad (6.53)$$

Since:

$$\left| \tfrac{1}{2}(A+D) \right| < 1 \qquad (6.54)$$

$\hookrightarrow$ physical solution

then:

$$\cos(K\Lambda) = \tfrac{1}{2}(A+D) \qquad (6.55)$$

$\hookleftarrow$ Dispersion relation

(6.50) into (6.45) yields...

$$\begin{bmatrix} a_0 \\ b_0 \end{bmatrix} = e^{inK\Lambda} \begin{bmatrix} a_n \\ b_n \end{bmatrix}$$

$$\Rightarrow \quad \begin{bmatrix} a_n \\ b_n \end{bmatrix} = e^{-inK\Lambda} \begin{bmatrix} a_0 \\ b_0 \end{bmatrix} \qquad (6.56)$$

(6.50) also provides (for $n = 0$)...

$$B b_0 = (e^{iK\Lambda} - A) a_0 \qquad (6.57)$$

(6.57) permits (6.56) to express the eigenvector, to within an arbitrary constant, as

$$\begin{bmatrix} a_n \\ b_n \end{bmatrix} = e^{-inK\Lambda} \begin{bmatrix} B \\ e^{iK\Lambda} - A \end{bmatrix} \qquad (6.58)$$

(6.55) may be written to express the explicit dependence of K on ω & k_y, as

$$K(\omega, k_y) = \Lambda^{-1} \cos^{-1}\left(\tfrac{1}{2}[A+D]\right) \qquad (6.59)$$

Let us consider this more closely:

- For $|\frac{1}{2}(A+D)| < 1$, K is real
- For $|\frac{1}{2}(A+D)| > 1$, K is complex

$\hookrightarrow$ see (6.52)

$$\hookrightarrow \quad K = \frac{m\pi}{\Lambda} + iK_i$$

Bloch wave is evanescent

$\Rightarrow$ Photonic Band Gaps ...

... where the photonic band edges are determined from

$$\left| \tfrac{1}{2}(A+D) \right| = 1 \qquad (6.60)$$

Now, is $E_x(z)$ of (6.21) really a Bloch wave? Consider its form in region I (layer n_1) of the n^{th} unit cell, using (6.56):

$$E_x(z) = e^{-inK\Lambda}\left\{ a_0 e^{ik_{1z}(n\Lambda - z)} + b_0 e^{-ik_{1z}(n\Lambda - z)} \right\}$$

$$= \left\{ \left[a_0 e^{-i(z-n\Lambda)} + b_0 e^{+i(z-n\Lambda)} \right] e^{iK(z-n\Lambda)} \right\} e^{-iKz} \qquad (6.61)$$

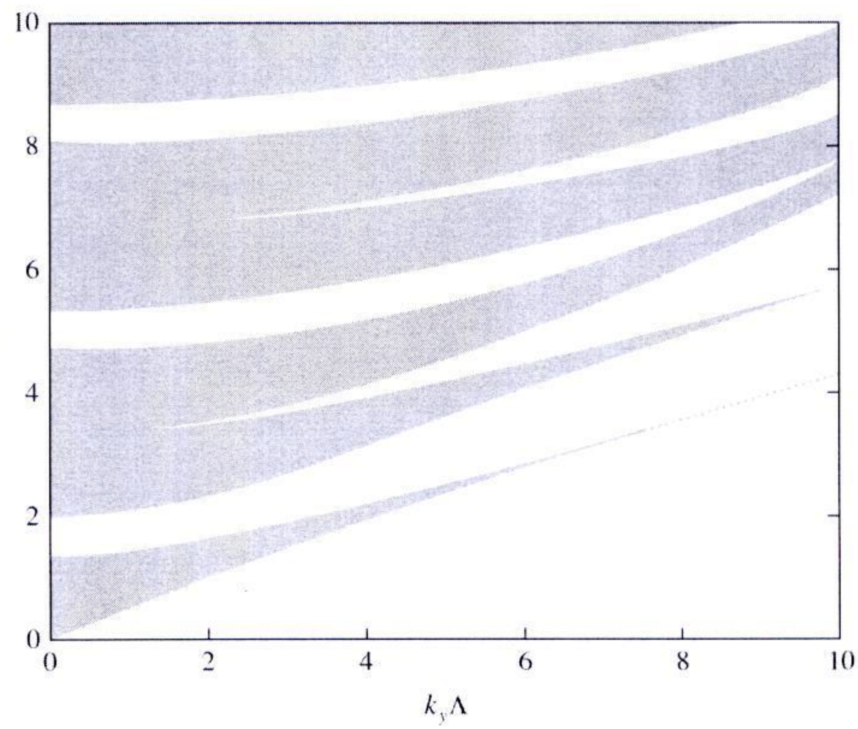
Clearly, (6.61) satisfies the Bloch theorem.

One may write the dispersion relation (6.55) for the TE and TM modes, using (6.36) & (6.39), and (6.41) & (6.44), respectively, as ...

$$\cos(K\Lambda) = \begin{cases} \cos(k_{1z}a)\cos(k_{2z}b) - \frac{1}{2}\left[\frac{k_{2z}}{k_{1z}} + \frac{k_{1z}}{k_{2z}}\right]\sin(k_{1z}a)\sin(k_{2z}b) & \rightarrow \text{TE} \\ \cos(k_{1z}a)\cos(k_{2z}b) - \frac{1}{2}\left[\frac{n_2^2 k_{1z}}{n_1^2 k_{2z}} + \frac{n_1^2 k_{2z}}{n_2^2 k_{1z}}\right]\sin(k_{1z}a)\sin(k_{2z}b) & \rightarrow \text{TM} \end{cases} \quad (6.62)$$

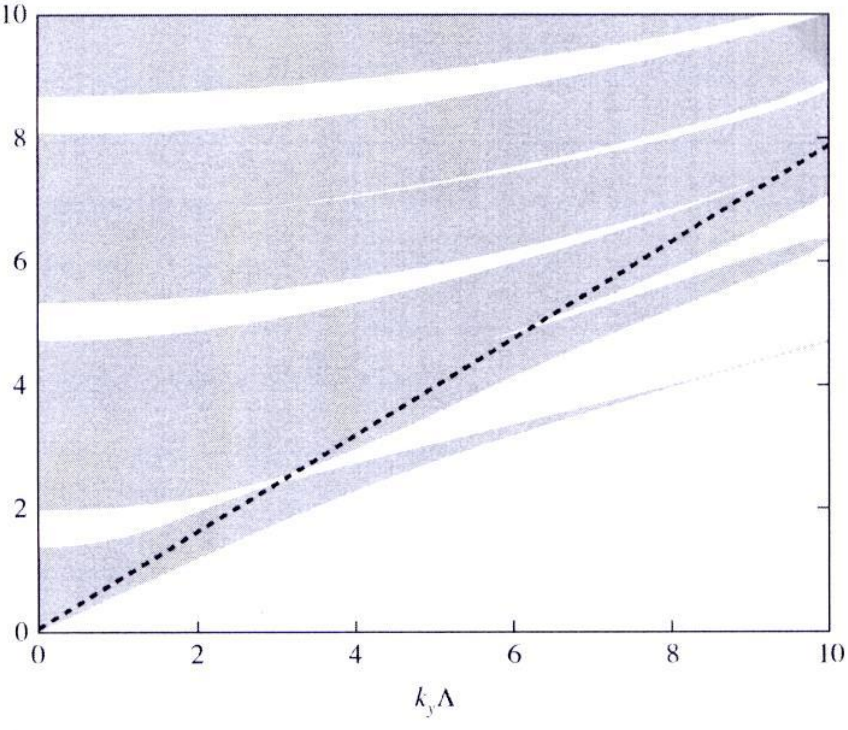
as illustration let us show the photonic band structure for TE & TM waves, choosing $n_1 = 1.45$, $n_2 = 2.65$, $a = \lambda_0/4n_1$, $b = \lambda_0/4n_2$ with $\lambda_0 = 1.5\,\mu m$. The grey bands denote propagating mode solutions.

"Photonic Bandstructure, TE mode"



Yariv & Yeh, 6th ed, p. 550.

"Photonic Bandstructure, TM mode"



Yariv & Yeh, 6th ed, p. 550

NB. The dashed line denotes

$$k_y = (\omega/c)\, n_2 \sin\theta_B \qquad (6.63)$$

where θ_B is the Brewster angle. Here, reflection vanishes for the TM mode, therefore no bandgaps can exist for such incident angles.