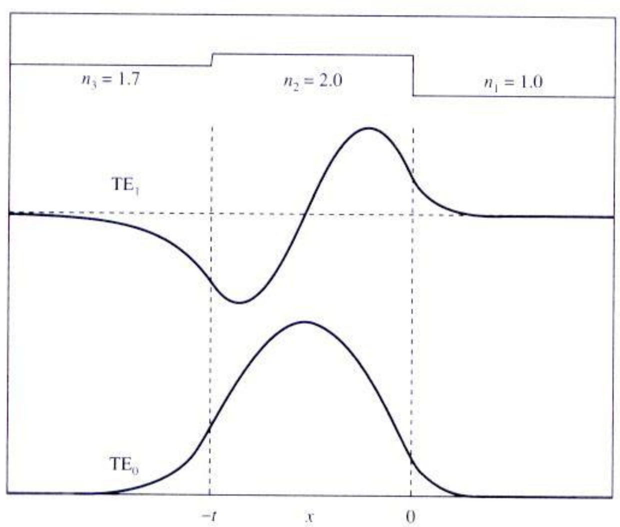
The orthonormalization condition may then be written as

$$\frac{1}{2}\int_{S}^{\infty}\left(\vec{\mathcal{E}}_m \times \vec{\mathcal{H}}_n^{*}\right)\cdot\hat{z}\, dS = \delta_{mn} \qquad (5.60)$$

Kronecker delta

Mode Confinement Factor

While the fields of the confined modes are evanescent into the cladding region ...



$n_2 = 2.0$
$n_3 = 1.7$
$n_1 = 1.0$

$t/\lambda = 1.0$

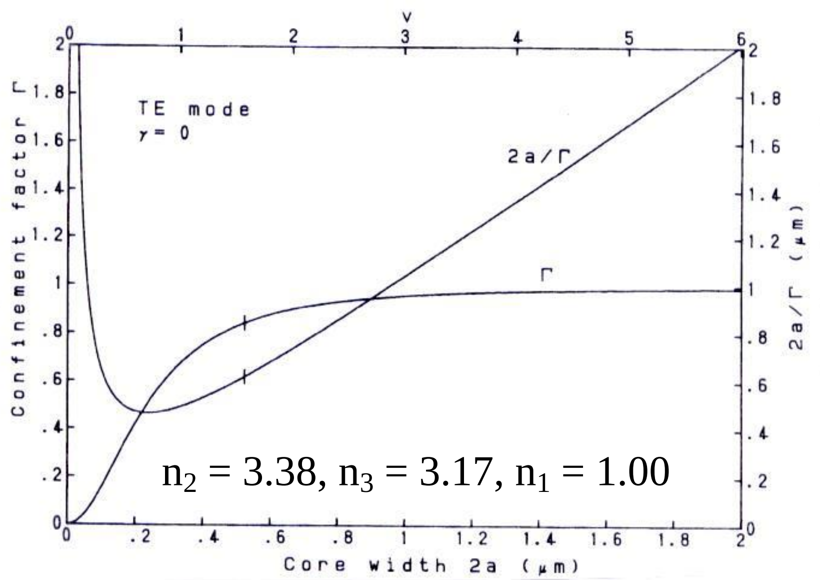
Yariv & Yeh, 6th ed, p. 123

... the time-averaged power flows in the z-direction

Let P be the power in the modal field $[\vec{E}, \vec{H}]$. The fraction of power flowing through any modal cross section m is

$$\Gamma_m = \frac{P_m}{P} = \frac{\frac{1}{2}\,\mathrm{Re}\int_m (\vec{E}\times\vec{H}^*)\cdot\hat{z}\, dS}{\frac{1}{2}\,\mathrm{Re}\int_S^{\infty} (\vec{E}\times\vec{H}^*)\cdot\hat{z}\, dS} \qquad (5.61)$$

Mode Confinement Factor ... if region m is the core



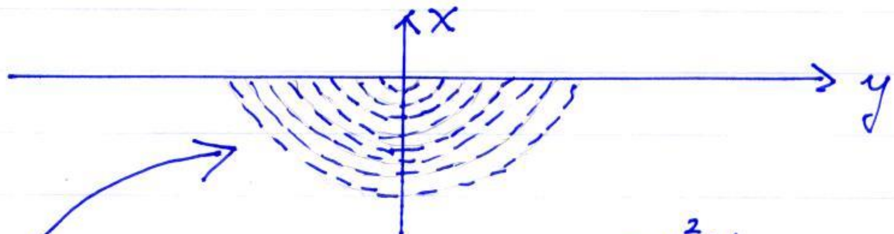
Okamoto, "Fundamentals of Optical Waveguides", 2ND ed., p.25

Two Dimensional Waveguides

Conventional geometries ...

(a) Channel $\longleftrightarrow$ diffused ...

- ion exchange
- thermal diffusion



$$n(x,y) \sim n_0 + (\Delta n) e^{-y^2/d_y} \operatorname{erfc}(x/d_x)$$

maximum index change

bulk refractive index

refractive index profile

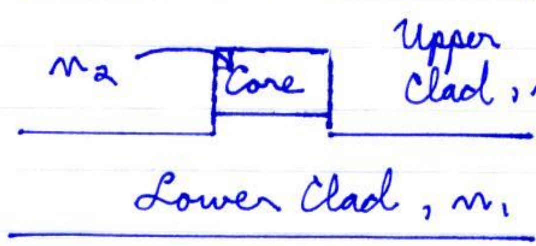
d_x - diffusion depth

d_y - lateral diffusion distance

$$\operatorname{erfc}(x) = 1 - \operatorname{erf}(x)$$

$$= \frac{2}{\sqrt{\pi}} \int_x^\infty e^{-t^2} dt$$

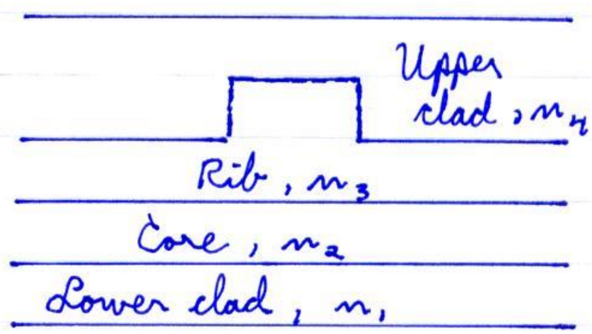
(b) Rectangular $\longleftrightarrow$ Buried core ...



$n(x, y)$

(c) Ridge ...

$n(x, y)$



Symmetry breaking prevents existence of purely TE or TM modes

Two classes of modes:

(1) HE_{pq} : H_x & E_y are dominant

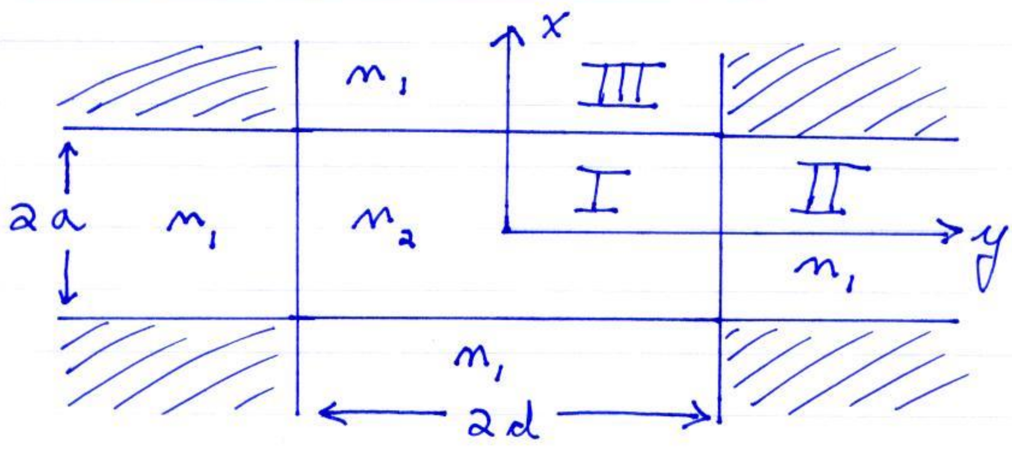
$\quad \hookrightarrow E^y_{(p+1)(q+1)}$ "Quasi-TE"

(2) EH_{pq} : E_x & H_y are dominant

$\quad \hookrightarrow E^x_{(p+1)(q+1)}$ "Quasi-TM"

Such modes may, in general, only be found numerically. There are, however, several analytic treatments that yield approximate solutions.

Marcatili Method (MM)



... assumes EM fields in "corners" can be neglected

Let us assume modal solutions of the form given by (5.3) & (5.4), and consider...

... HE modes.

$$MM \longrightarrow H_y = 0 \tag{5.62}$$

$$(H_x = 0, \text{ for } EH)$$

(5.62) into (5.5) through (5.10) yields

$$\frac{\partial^2 H_x}{\partial y^2} + \frac{\partial^2 H_x}{\partial x^2} + (k^2 n^2 - \beta^2) H_x = 0 \tag{5.63}$$

$$E_x = \frac{1}{\omega \varepsilon_0 n^2 \beta} \frac{\partial^2 H_x}{\partial x \partial y} \tag{5.64}$$

$$E_y = \frac{\omega \mu_0}{\beta} H_x + \frac{1}{\omega \varepsilon_0 n^2 \beta} \frac{\partial^2 H_x}{\partial y^2} \tag{5.65}$$

$$E_z = \frac{-j}{\omega \varepsilon_0 n^2} \frac{\partial H_x}{\partial y} \tag{5.66}$$

$$H_z = \frac{-j}{\beta} \frac{\partial H_x}{\partial x} \tag{5.67}$$

Since the waveguide is symmetric with respect to the x & y axes, we only need to consider regions I, II, & III. The solution to (5.63) must have the general form

$$H_x(x,y) = \begin{cases} A\cos(k_x x - \phi)\cos(k_y y - \psi), & \mathrm{I} \\ A\cos(k_x x - \phi)\cos(k_y d - \psi)e^{-\gamma_y (y-d)}, & \mathrm{II} \\ A\cos(k_x a - \phi)\cos(k_y y - \psi)e^{-\gamma_x (x-a)}, & \mathrm{III} \end{cases} \tag{5.68}$$

where (5.68) into (5.63) yields

$$-k_x^2 - k_y^2 + k^2 n_2^2 - \beta^2 = 0 \ , \ \mathrm{I} \tag{5.69}$$

$$-k_x^2 + \gamma_y^2 + k^2 n_1^2 - \beta^2 = 0 \ , \ \mathrm{II} \tag{5.70}$$

$$\gamma_x^2 - k_y^2 + k^2 n_1^2 - \beta^2 = 0 \ , \ \mathrm{III} \tag{5.71}$$

and
$$\phi = p\,\pi/2 \ , \ p = 0, 1, \ldots \tag{5.72}$$

$$\psi = q\,\pi/2 \ , \ q = 0, 1, \ldots \tag{5.73}$$

We must now apply the boundary conditions

↳ tangential components of $\vec{E}$ & $\vec{H}$ are continuous across the boundaries ...

For $x = a$: E_z is continuous; therefore, from (5.66)

$$\frac{1}{n^2}\frac{\partial H_x}{\partial y} \text{ is continuous} \tag{5.74}$$

For $y = d$: H_z is continuous; therefore, from (5.67)

$$\frac{\partial H_x}{\partial x} \text{ is continuous} \tag{5.75}$$

The dispersion relations are obtained from (5.68), by applying (5.74) & (5.75) as

$$k_y d = q\,\pi/2 + \tan^{-1}\left(\frac{n_2^2\,\gamma_y}{n_1^2\,k_y}\right) \qquad (5.76)$$

$$k_x a = p\,\pi/2 + \tan^{-1}\left(\frac{\gamma_x}{k_x}\right) \qquad (5.77)$$

The wave numbers are also related, through (5.69) & (5.70) as

$$\gamma_y^2 = k^2(n_2^2 - n_1^2) - k_y^2, \qquad (5.78)$$

and (5.69) & (5.71) as

$$\gamma_x^2 = k^2(n_2^2 - n_1^2) - k_x^2 \qquad (5.79)$$

(5.78) & (5.76) permits determination of k_y;
(5.79) & (5.77) " " " k_x ...
... and β is then determined via (5.69) as

$$\beta^2 = k n_2^2 - (k_x^2 + k_y^2) \qquad (5.80)$$

The analysis for EH modes proceeds in a similar manner.

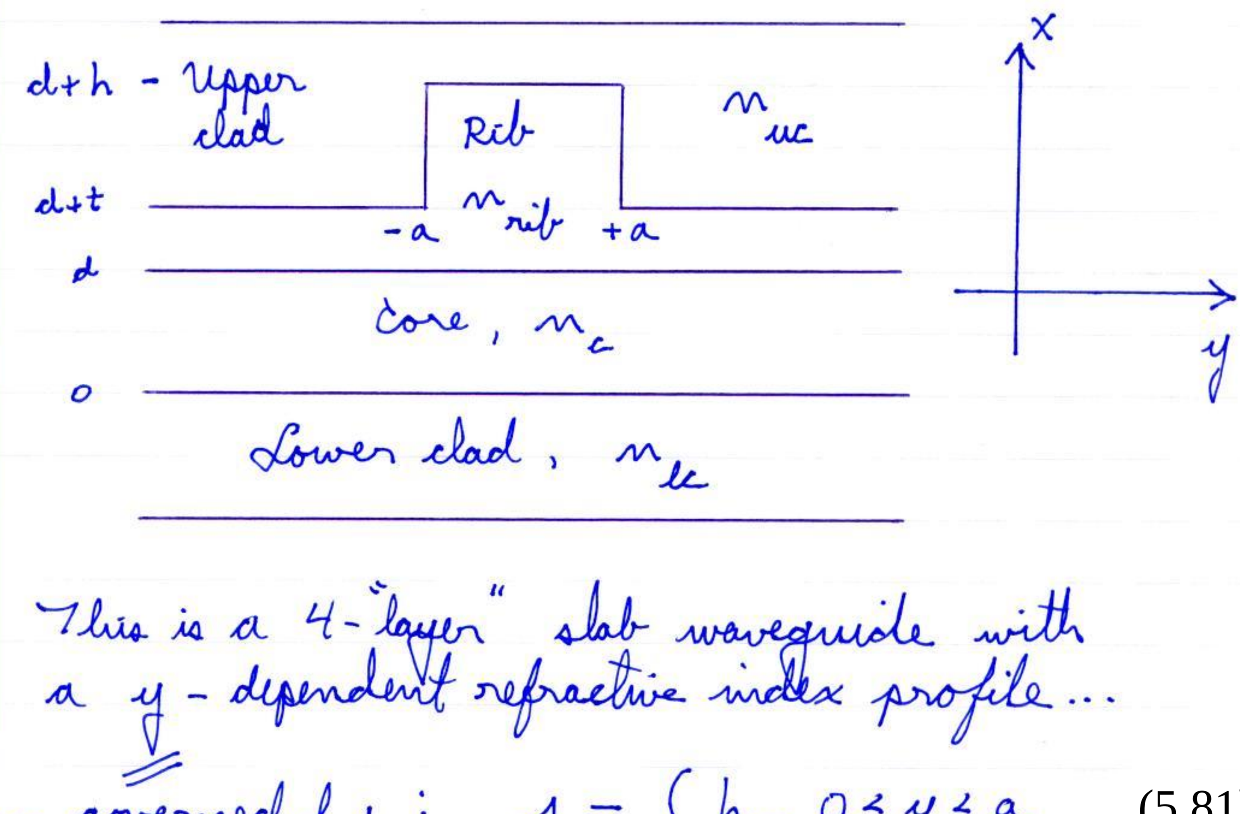
Effective Index Method

another analytic approach to generate an approximate solution

Let us illustrate this method using the

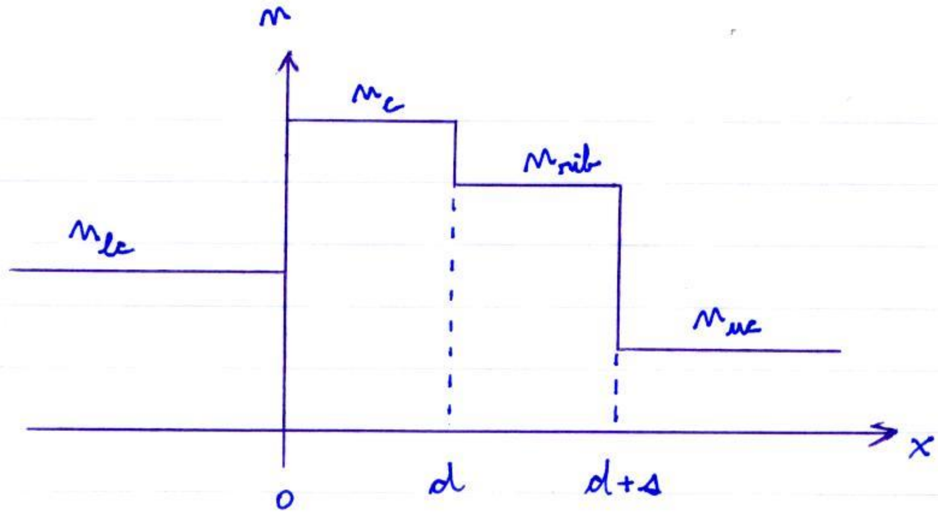
$$E^{y}_{(p+1)(q+1)} \text{ mode}$$

in a ridge waveguide



This is a 4-"layer" slab waveguide with a y-dependent refractive index profile...

... governed by : $$s = \begin{cases} h, & 0 \le y \le a \\ t, & |y| > a \end{cases} \quad (5.81)$$



as noted earlier, this mode is determined via (5.63):

$$\frac{\partial^2 H_x}{\partial x^2} + \frac{\partial^2 H_x}{\partial y^2} + (k^2 n^2 - \beta^2) H_x = 0$$

IF we can assume separation of variables

$$H_x(x, y) = X(x)\, Y(y) \qquad (5.82)$$

then (5.82) into (5.63) yields...

$$\frac{1}{X}\frac{d^2 X}{dx^2} + \frac{1}{Y}\frac{d^2 Y}{dy^2} + [k^2 n^2(x,y) - \beta^2] = 0 \qquad (5.83)$$

To separate (5.83), assume an effective index profile $n_{eff}(y)$. Adding & subtracting n_{eff} in (5.83) permits it to be separated as

$$\frac{1}{X}\frac{d^2X}{dx^2} + [k^2n^2(x,y) - k^2n_{eff}^2(y)] = 0 \qquad (5.84)$$

$$\frac{1}{Y}\frac{d^2Y}{dy^2} + [k^2n_{eff}^2(y) - \beta^2] = 0 \qquad (5.85)$$

(5.84) is first solved for $n_{eff}(y)$ in a manner similar to that described for the asymmetric 3-layer waveguide. One finds that the dispersion relation is given by

$$\sin(\kappa d - 2\phi) = \sin(\kappa d)\, e^{-2(\sigma s + \psi)} \qquad (5.86)$$

where $\phi = \tan^{-1}(\sigma/\kappa)$

$$\psi = \tanh^{-1}(\sigma/\gamma)$$

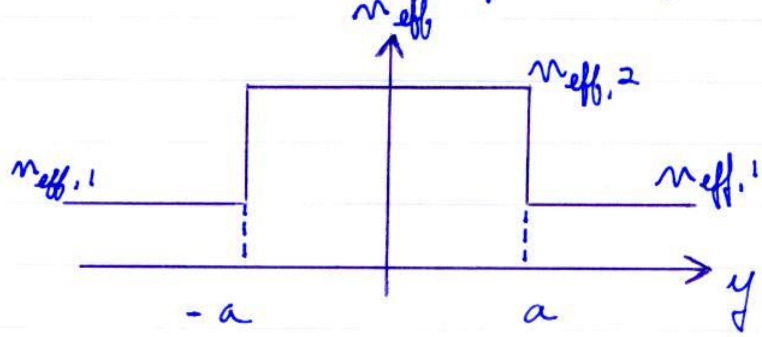
$$\kappa = k\sqrt{n_c^2 - n_{eff}^2} \qquad (5.87)$$

$$\sigma = k\sqrt{n_{eff}^2 - n_{lc}^2}$$

$$\gamma = k\sqrt{n_{eff}^2 - n_{uc}^2}$$

This follows from (5.67), the continuity of H_z at $x = 0$, d & $d+s$.

Solving (5.86) for $s = h$, using (5.87), gives $n_{eff,2}$ while for $s = t$, one gets $n_{eff,1}$



Now (5.85) may be solved as a 3-layer symmetrical waveguiding problem.

Careful! Note that the boundary condition on (5.85) is, from (5.66):

$$E_z \propto \frac{1}{n^2}\frac{\partial H_x}{\partial y} \Rightarrow \frac{1}{n^2}\frac{dY}{dy} \text{ is continuous at } y = \pm a$$

The dispersion relation is then found as

$$u \tan(u) = \left(\frac{n_{eff,2}^2}{n_{eff,1}^2}\right) w \qquad (5.88)$$

where:
$$u = ka\sqrt{n_{eff,2}^2 - (\beta/k)^2} \qquad (5.89)$$

$$w = ka\sqrt{(\beta/k)^2 - n_{eff,1}^2} \qquad (5.90)$$

Surface Plasmon Waveguides

From our earlier discussion of the Drude mode, we understand that metals behave like plasmas at optical frequencies.

If: $\omega < \omega_p$ ← plasma frequency, (2.87)

then: $\varepsilon_e(\omega) = \varepsilon_0 \left(1 - \omega_p^2/\omega^2\right) < 0$

ε_e ← effective permittivity, (2.86)

⇒ wave is strongly attenuated in the metal

at $\omega = \omega_p$, $\varepsilon_e(\omega) = 0$ and hence

$k = \omega\sqrt{\mu\,\varepsilon_e(\omega)}$ vanishes

⇒ $\lim_{\omega \to \omega_p^+}$: the propagating mode vanishes

The medium response is, from (1.1)

$$\vec{D} = \varepsilon_e(\omega)\vec{E} = \varepsilon_0 \vec{E} + \vec{P} = 0 \tag{5.91}$$

$$\Rightarrow \quad \vec{P} = -\varepsilon_0 \vec{E} \tag{5.92}$$

the induced polarization is opposite to the $\vec{E}$-field

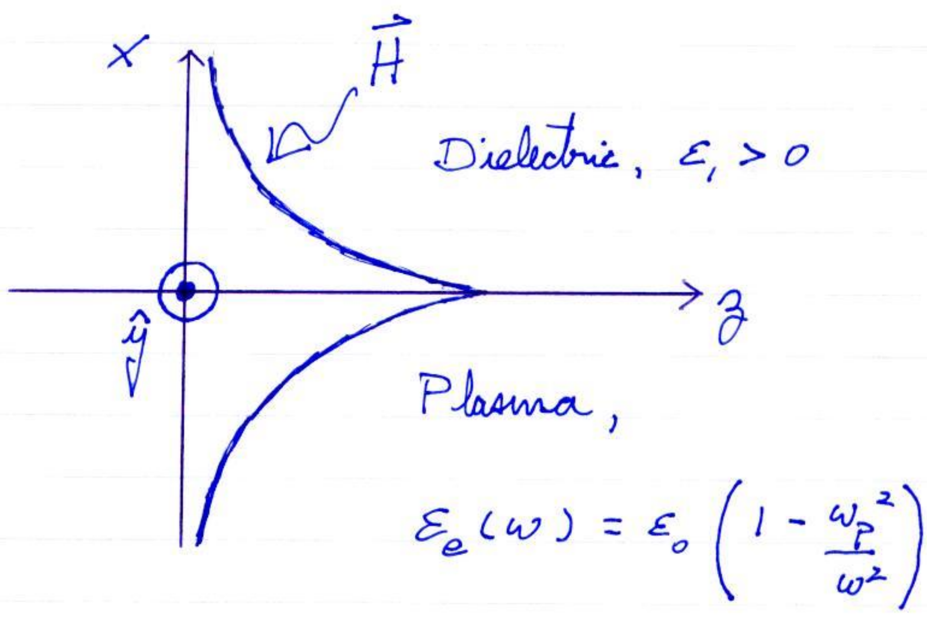
Surface Plasmon Mode - Single Interface

Consider the interface between a dielectric and a plasma. Is there a guided mode near the surface ...

... exponentially decaying for $|x| \to \infty$?

Yes.
$$\vec{H} = \hat{y} H_y = \hat{y} H_0 e^{ik_z z} \begin{cases} e^{-\alpha_1 x}, & x \geq 0 \\ e^{\alpha_2 x}, & x < 0 \end{cases} \tag{5.93}$$

TM mode



The wave equation in each region gives

$$-\alpha_1^2 + k_z^2 = \omega^2 \mu_0 \varepsilon_1 \tag{5.94}$$

$$-\alpha_2^2 + k_z^2 = \omega^2 \mu_0 \varepsilon_e(\omega) \tag{5.95}$$

The electric field may be obtained as

$$\vec{E} = \begin{cases} (i\omega\varepsilon_1)^{-1}(\alpha_1 \hat{z} + ik_z \hat{x}) H_0 \exp(-\alpha_1 x + ik_z z), & x \geq 0 \\ (i\omega\varepsilon_e)^{-1}(-\alpha_2 \hat{z} + ik_z \hat{x}) H_0 \exp(\alpha_2 x + ik_z z), & x \leq 0 \end{cases} \tag{5.96}$$

The boundary conditions assert that E_z must be continuous across the interface, hence

$$\frac{\alpha_1}{\varepsilon_1} = -\frac{\alpha_2}{\varepsilon_e} \tag{5.97}$$

(5.97) into (5.94) & (5.95) yields...

$$\alpha_1 = \omega \sqrt{\frac{-\mu_0 \varepsilon_1^2}{\varepsilon_1 + \varepsilon_e}} \tag{5.98}$$

$$\alpha_2 = \omega \sqrt{\frac{-\mu_0 \varepsilon_e^2}{\varepsilon_1 + \varepsilon_e}} \tag{5.99}$$

$$k_z = \omega \sqrt{\frac{\mu_0 \varepsilon_1 \varepsilon_e}{\varepsilon_1 + \varepsilon_e}} \tag{5.100}$$

(5.93) & (5.96) into (1.68) gives the power flow density in the propagation direction as

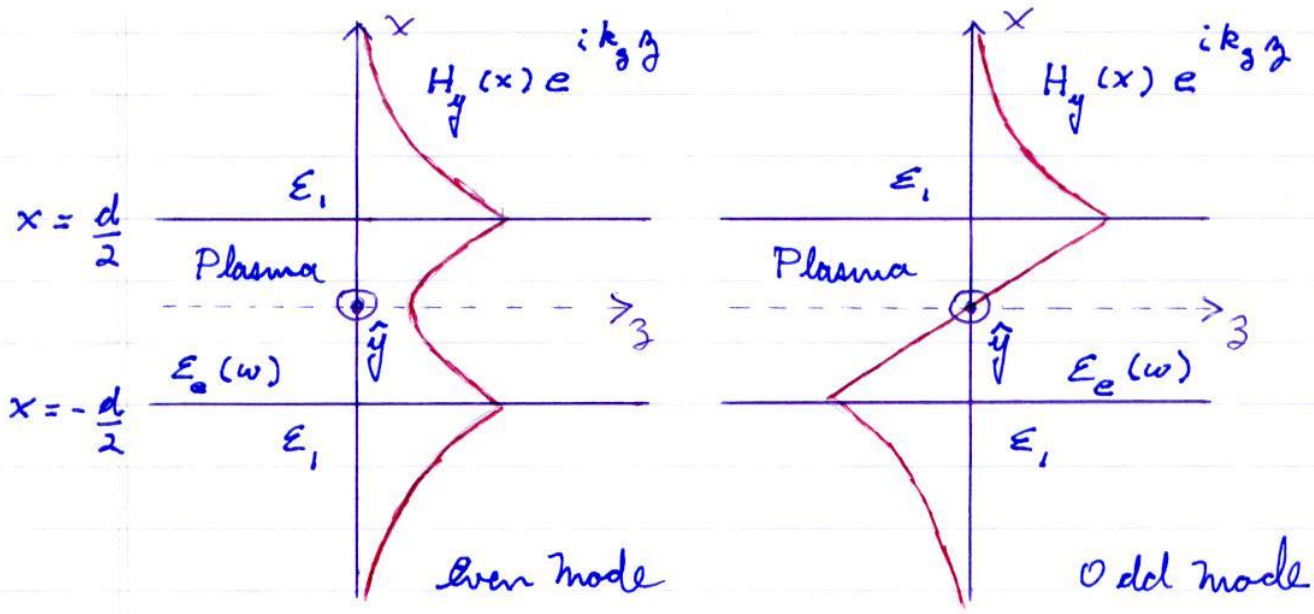
$$P = \frac{1}{2} \mathrm{Re} (\vec{E} \times \vec{H}^*) \cdot \hat{z}$$

$$= \begin{cases} k_z (2\omega\varepsilon_1)^{-1} |H_0|^2 e^{-2\alpha_1 x} & , \ x \geq 0 \\ k_z (2\omega\varepsilon_e)^{-1} |H_0|^2 e^{2\alpha_2 x} & , \ x \leq 0 \end{cases} \tag{5.101}$$

Surface Plasmon Modes in a Metallic Slab

... in a symmetric dielectric ε_1

$\hookrightarrow$ both even & odd TM modes



NB. $\omega < \omega_p \Rightarrow \varepsilon_e < 0$

TM even modes:

↳ Solutions are of the form ...

$$\vec{H} = \hat{y}\, e^{ik_z z} \begin{cases} C_0\, e^{-\alpha_1 (x - d/2)} & , \quad x \geq \frac{1}{2} d \\ C_1 \cosh(\alpha_2 x) & , \quad |x| \leq \frac{1}{2} d \\ C_0\, e^{\alpha_1 (x + d/2)} & , \quad x \leq -\frac{1}{2} d \end{cases} \qquad (5.102)$$

(5.102) into the relevant wave equation yields, as before

$$-\alpha_1^2 + k_z^2 = \omega^2 \mu \varepsilon_1 \qquad (5.103)$$

$$-\alpha_2^2 + k_z^2 = \omega^2 \mu \varepsilon_c \qquad (5.104)$$

The boundary conditions are the continuity of the tangential components E_z & H_y at $x = \pm d/2$

$$E_z = \frac{-1}{i\omega\varepsilon} \frac{\partial H_y}{\partial x}$$

These yield

$$C_0 = C_1 \cosh(\alpha_2 d/2) \qquad (5.105)$$

$$-(\alpha_1/\varepsilon_1)\, C_0 = C_1 (\alpha_2/\varepsilon_c) \sinh(\alpha_2 d/2) \qquad (5.106)$$

The ratio of (5.106) & (5.105) yields...

$$\alpha_1 = -(\varepsilon_1/\varepsilon_e)\,\alpha_2 \tanh(\alpha_2 d/2) \qquad (5.106)$$

(5.103) – (5.104) yields...

$$\alpha_2^2 - \alpha_1^2 = \omega^2 \mu_0 (\varepsilon_1 - \varepsilon_e) \qquad (5.106)$$

Thus, (5.107) & (5.108) may be solved to find α_1, α_2

TM Odd modes:

... may be determined in a similar manner