Ch.5 Guided Waves

5.1 Introduction

- finite beams spread during propagation in free space
  $\hookrightarrow$ diffraction

Confinement? Lenses, gradient index media...

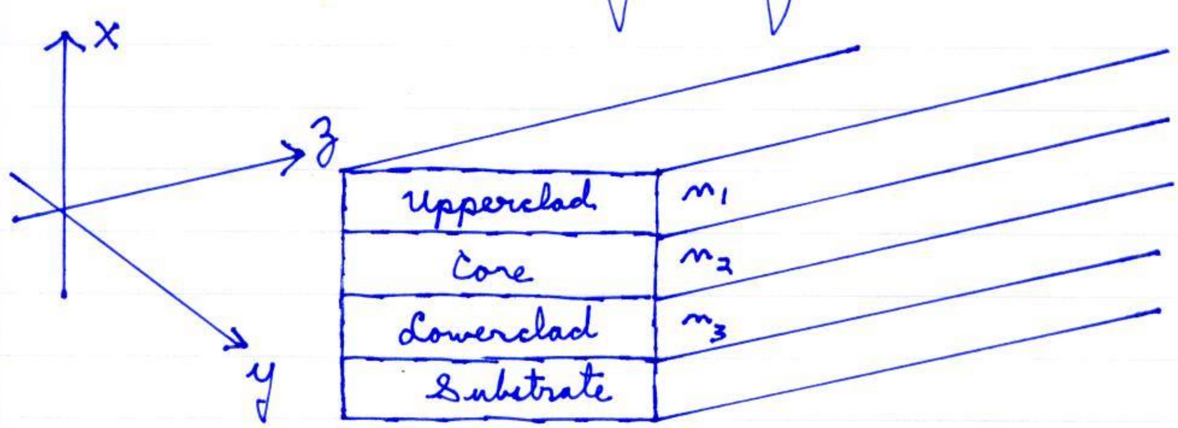
Consider propagation in guiding structures

→ Confinement : "total internal reflection"

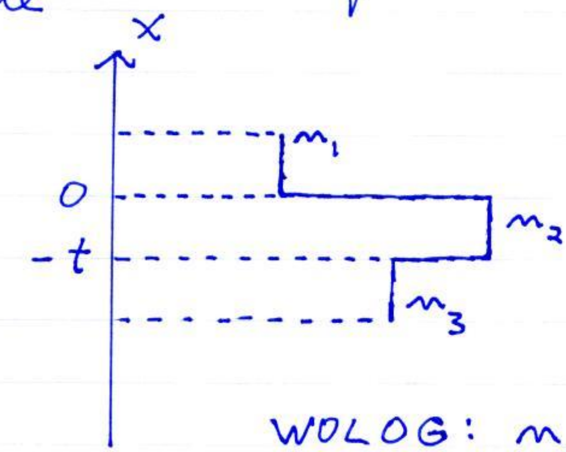
→ "waveguide modes"

5.2 Slab Waveguides

Consider a planar waveguiding structure ...



... with the following refractive index profile



WOLOG: $n_1 < n_3 < n_2$

note that the medium is homogeneous in each of the layers. Therefore Maxwell's curl equations (1.3) & (1.4) can be written as

$$\vec{\nabla} \times \vec{E} = -\mu_o \frac{\partial \vec{H}}{\partial t} \qquad (5.1)$$

$$\vec{\nabla} \times \vec{H} = \varepsilon_o n^2(x) \frac{\partial \vec{E}}{\partial t} \qquad (5.2)$$

Let us assume monochromatic solutions to (5.1) & (5.2) of the form

$$\mathbf{E}(\mathbf{r},t) = \vec{E}_m(x,y)\, e^{i(\omega t - \beta z)} \qquad (5.3)$$

$$\mathbf{H}(\mathbf{r},t) = \vec{H}_m(x,y)\, e^{i(\omega t - \beta z)} \qquad (5.4)$$

(5.3) & (5.4) into (5.1) yields...

$$\frac{\partial E_z}{\partial y} + i\beta E_y = -i\omega\mu_o H_x \qquad (5.5)$$

$$-\frac{\partial E_z}{\partial x} - i\beta E_x = -i\omega\mu_o H_y \qquad (5.6)$$

$$\frac{\partial E_y}{\partial x} - \frac{\partial E_x}{\partial y} = -i\omega\mu_o H_z \qquad (5.7)$$

... while (5.3) & (5.4) into (5.2) yields:

$$\frac{\partial H_z}{\partial y} + i\beta H_y = i\omega \varepsilon_0 n^2 E_x \tag{5.8}$$

$$-\frac{\partial H_z}{\partial x} - i\beta H_x = i\omega \varepsilon_0 n^2 E_y \tag{5.9}$$

$$\frac{\partial H_y}{\partial x} - \frac{\partial H_x}{\partial y} = i\omega \varepsilon_0 n^2 E_z \tag{5.10}$$

By symmetry, $\vec{E}$ & $\vec{H}$ can have no dependence on y.

$$\Rightarrow \quad \frac{\partial \vec{E}}{\partial y} = 0 \tag{5.11}$$

$$\frac{\partial \vec{H}}{\partial y} = 0 \tag{5.12}$$

(5.11) & (5.12) into (5.5) through (5.10) yield two independent modes, denoted TE & TM.

The TE mode satisfies

$$\frac{d^2 E_y}{dx^2} + (k^2 n^2 - \beta^2) E_y = 0 \tag{5.13}$$

where $$H_x = \frac{-\beta}{\omega\mu_0} E_y \tag{5.14}$$

$$H_z = \frac{i}{\omega\mu_0} \frac{dE_y}{dx} \tag{5.15}$$

and $$E_x = E_z = H_y = 0 \tag{5.16}$$

We also note, from (1.11) & (1.12), that the tangential components E_y & H_z should be continuous across the planar interfaces.

Terminology:

Plane of incidence = plane of propagation

$\rightarrow$ xz plane

identified by symmetry breaking

TE: transverse electric

$\hookrightarrow$ $\vec{E}$-field is transverse to plane of incidence

TM: transverse magnetic

$\hookrightarrow$ $\vec{H}$-field is transverse to plane of incidence

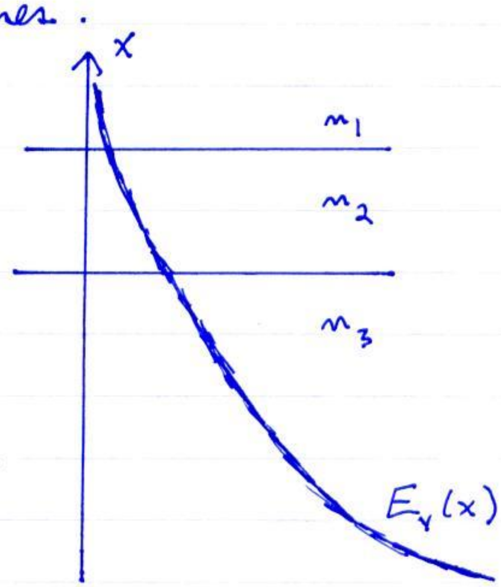
The TM mode satisfies equations similar to (5.13) through (5.16).

Let us consider the TE case. What is the physical nature of the solution? We know, as per (5.3), that as each layer is a homogeneous medium, the solution to (5.13) will involve relating the particular layer solution forms to one another by application of the boundary conditions.

What are the possibilities? In (5.13), consider

$$k^2 n^2 - \beta^2 = (n\omega/c)^2 - \beta^2$$

There are four frequency regimes.



(a) For $\beta > n_2\omega/c$:

(5.13) $\Rightarrow \dfrac{1}{E}\dfrac{d^2E_y}{dx^2} > 0$

ie. $E_y(x) \sim e^{-\gamma x}$ in all layers

BUT... as $x \to -\infty$, $E_y \to \infty$

$\therefore$ unphysical solution — unrealizable.

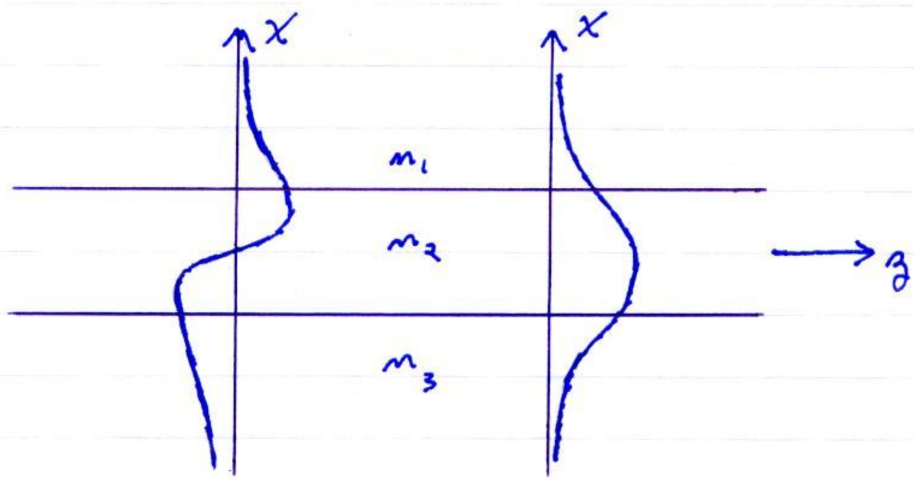
(b) For $n_3\omega/c < \beta < n_2\omega/c$:

(5.13) $\Rightarrow \quad \frac{1}{E}\frac{d^2E_y}{dx^2} \begin{cases} < 0, & \text{for } -t \leq x \leq 0 \\ > 0, & \text{otherwise} \end{cases}$

ie. E_y is sinusoidal in the core
E_y is exponential in the cladding

$\therefore$ Both finiteness and boundary conditions can be satisfied

$\rightarrow$ valid solutions

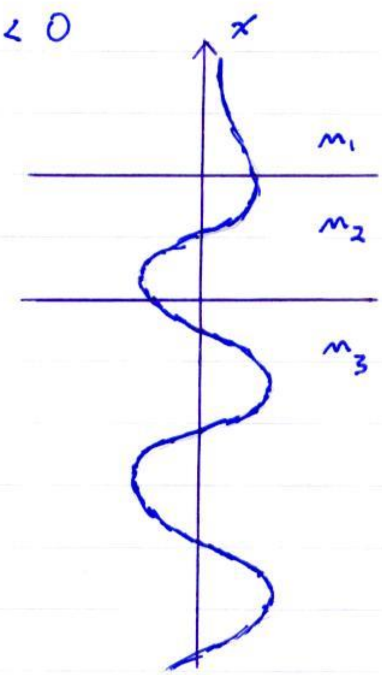


"Bound modes"

└ energy propagating in z-direction

(c) For $n_1 \omega/c < \beta < n_3 \omega/c$:

(5.13) $\Rightarrow \frac{1}{E}\frac{d^2E_y}{dx^2} \begin{cases} > 0, & x > 0 \\ < 0, & x < 0 \end{cases}$



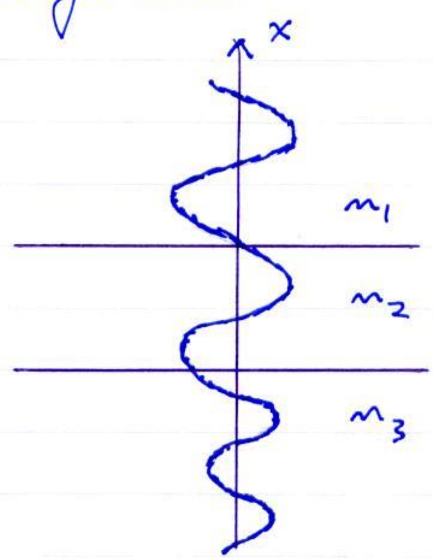
$\Rightarrow$ exponential in the upper cladding, sinusoidal elsewhere

$\hookrightarrow$ "leaky" mode

$\hookrightarrow$ Substrate radiation mode

(d) For $0 < \beta < n_1 \omega/c$

(5.13) $\Rightarrow \frac{1}{E}\frac{d^2E_y}{dx^2} < 0$ everywhere



$\Rightarrow$ sinusoidal everywhere

$\hookrightarrow$ radiation mode

ie. unbound

Guided TE modes: $\vec{E} \to E_y$ only

$$E_y(x,y,z,t) = E_m(x)\, e^{i(\omega t - \beta_m z)} \qquad (5.17)$$

wavefunction of the m^{th} mode

propagation constant

The most general form for $E_m(x)$ is

$$E_m(x) = \begin{cases} A e^{-q x}, & x \geq 0 \\ B \cos(hx) + C \sin(hx), & -t \leq x \leq 0 \\ D e^{p(x+t)}, & x \leq -t \end{cases} \qquad (5.18)$$

(5.18) into (5.13) yields...

$$h^2 = (n_2 \omega/c)^2 - \beta^2 \qquad (5.19)$$

$$q^2 = \beta^2 - (n_1 \omega/c)^2 \qquad (5.20)$$

$$p^2 = \beta^2 - (n_3 \omega/c)^2 \qquad (5.21)$$

We now apply our boundary conditions at the layer interfaces

(1.11) $\Rightarrow E_y$ is continuous ...

(1.12) $\Rightarrow H_z$ is continuous ...

From (1.11), using (5.18):

$$x = 0: \quad A = B \tag{5.22}$$

$$x = -t: \quad B\cos(ht) - C\sin(ht) = D \tag{5.23}$$

From (1.12), using (5.18) into (5.15):

$$x = 0: \quad -gA = hC$$

$$\Rightarrow C = -\frac{g}{h}A \tag{5.24}$$

$$x = -t: \quad hB\sin(ht) + hC\cos(ht) = pD \tag{5.25}$$

(5.22) & (5.24) into (5.23) yield...

$$D = A\{\cos(ht) + gh^{-1}\sin(ht)\} \tag{5.26}$$

Thus (5.22), (5.24), (5.26) into (5.18) yield...

$$E_m(x) = A \begin{cases} e^{-gx} & , x \geq 0 \\ \cos(hx) - gh^{-1}\sin(hx) & , -t \leq x \leq 0 \\ [\cos(ht) + gh^{-1}\sin(ht)]e^{p(x+t)} & , x \leq -t \end{cases} \tag{5.27}$$

The mode condition follows from (5.25) & (5.26):

$$h\sin(ht) - g\cos(ht) = p\{\cos(ht) + gh^{-1}\sin(ht)\}$$

The dispersion relation (mode condition) may be written as

$$\tan(ht) = \frac{h(p+q)}{h^2 - pq} \tag{5.28}$$

β must satisfy this condition

For a sufficiently large "t", there will be a finite number of solutions for $\beta \leftarrow \beta_m$

When do guided modes "disappear"?

$\longrightarrow$ Reduce ω until p vanishes

(p will vanish before q because $n_3 > n_1$)

From (5.21): $\beta = n_3 \omega / c = k_0 n_3$

In (5.28), $p = 0$ gives

$$\tan(ht) = q/h$$

$$\Rightarrow ht = \tan^{-1}(q/h) + m\pi \tag{5.29}$$

mode cutoff condition

at cutoff:

$$(5.21) \rightarrow \quad \beta = k_0 n_3 \qquad (5.30)$$

$$(5.20) \rightarrow \quad q = k_0 \sqrt{n_3^2 - n_1^2} \qquad (5.31)$$

$$(5.19) \rightarrow \quad h = k_0 \sqrt{n_2^2 - n_3^2} \qquad (5.32)$$

(5.31) & (5.32) into (5.29) yield...

$$\underbrace{k_0}_{\omega_m/c} t \sqrt{n_2^2 - n_3^2} = \tan^{-1}\left(\frac{n_3^2 - n_1^2}{n_2^2 - n_3^2}\right)^{1/2} + m\pi \qquad (5.33)$$

mode cutoff frequencies

NB. For a symmetric waveguide, (5.33) reduces to

$$\omega_m = \frac{m\pi c}{t\sqrt{n_{core}^2 - n_{clad}^2}} \qquad (5.34)$$

where $n_{core} = n_2$, $n_{clad} = n_1 = n_3$

$$\Rightarrow \frac{\omega_c}{2\pi} = \frac{c}{2t \cdot NA} \qquad (5.35)$$

$$NA = \sqrt{n_2^2 - n_3^2} \qquad (5.36)$$

numerical aperture

Hence, for a symmetric waveguide

$$\omega_m = m\,\omega_c \tag{5.37}$$

NB. another useful parameter is the "waveguide delta" ← the refractive index difference between core & cladding

$$\Delta = \frac{n_{core}^2 - n_{clad}^2}{2 n_{core}}$$

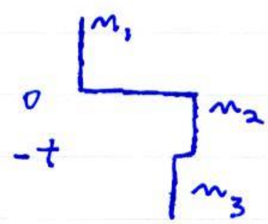
$$\approx \frac{n_{core} - n_{clad}}{n_{core}} \tag{5.38}$$

The mode cutoff frequencies for the asymmetric waveguide, (5.33), may be rewritten as a normalized angular frequency using (5.35):

$$\frac{\omega_m}{\omega_c} = m + \pi^{-1}\tan^{-1}\left(\frac{n_3^2 - n_1^2}{n_2^2 - n_3^2}\right)^{1/2} \tag{5.39}$$

NB. $0 \le \tan^{-1}(\cdots) < \pi/2$

Consider $n_2 \gtrsim n_3 \gg n_1$



$\Rightarrow \tan^{-1}(\cdots) \rightarrow \pi/2$

(5.39) then yields

$$\frac{\omega_m}{\omega_c} \approx m + \tfrac{1}{2} \qquad (5.40)$$

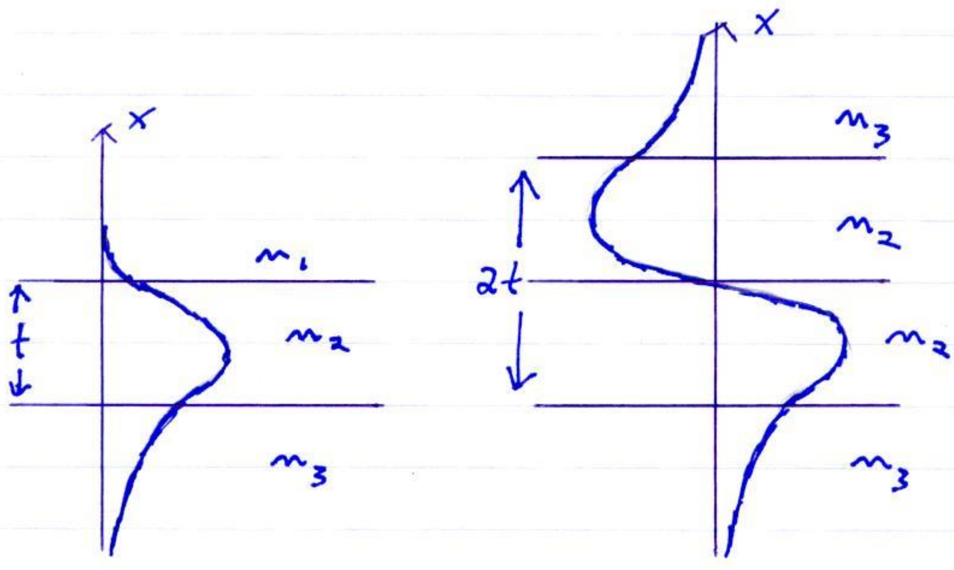
From (5.35):

$$\frac{\omega_m}{\omega_c} = k_0 c \left\{ \frac{2t\sqrt{n_2^2 - n_3^2}}{2\pi c} \right\}$$

$$= \pi^{-1} k_0 t \sqrt{n_2^2 - n_3^2} \qquad (5.41)$$

Equating (5.40) & (5.41)...

$$k_0 (2t)\sqrt{n_2^2 - n_3^2} \approx (2m+1)\pi \qquad (5.42)$$



⇒ Cutoff condition for TE_m mode (asymmetric, t) is the same as the equivalent "symmetric 2t" cutoff → TE_{2m+1} mode

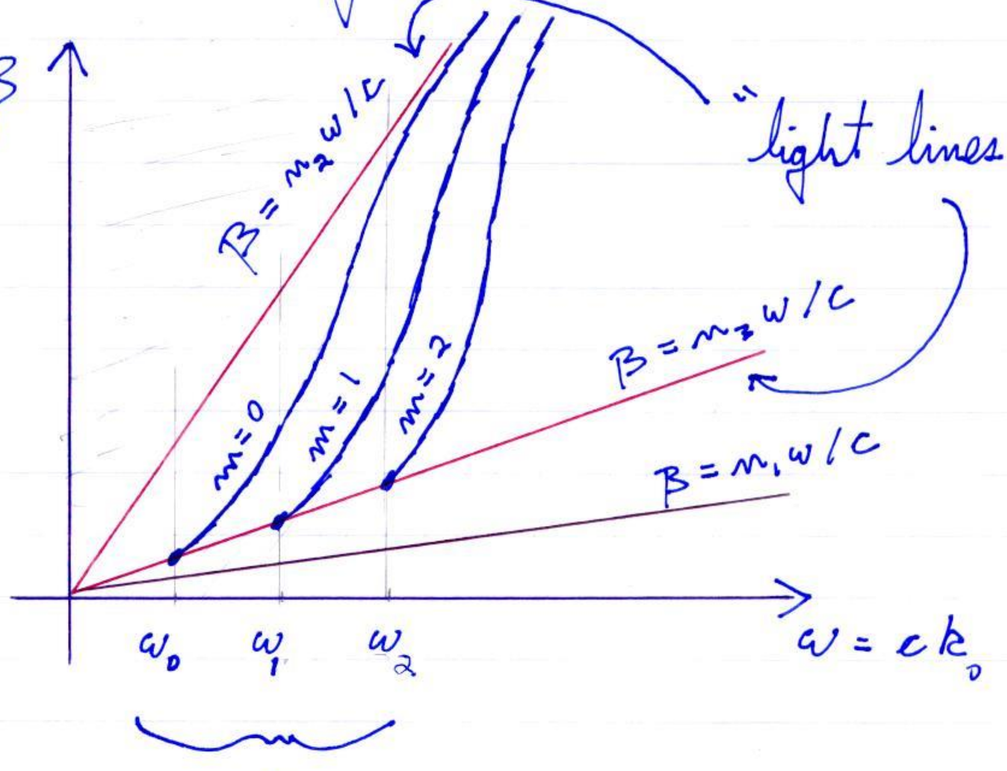
NB. By convention, the normalized frequency V is given by

$$V = \frac{1}{2} k_0 t \sqrt{n_2^2 - n_3^2} \qquad (5.43)$$

↑ the "V-number"

→ a metric for the single mode limit.

The dispersion relations for an arbitrary asymmetric waveguide behave as below:



↑ given by (5.39)

NB. We require $\omega > \omega_0$ for the existence of at least one mode. Using (5.33) & (5.43)...

$$V > \frac{1}{2}\tan^{-1}\left(\frac{n_3^2 - n_1^2}{n_2^2 - n_3^2}\right)^{1/2} \qquad (5.44)$$

NB. For arbitrary ω, $(m+1)$ guided TE modes exist if

$$m\frac{\pi}{2} + \frac{1}{2}\tan^{-1}(\cdots)^{1/2} < V < (m+1)\frac{\pi}{2} + \frac{1}{2}\tan^{-1}(\cdots)^{1/2} \qquad (5.45)$$

NB. For single-moded operation, choice of ω, t, and Δ is constrained such that, from (5.45), the TE_0 mode must satisfy

$$\frac{1}{2}\tan^{-1}\left(\frac{n_3^2 - n_1^2}{n_2^2 - n_3^2}\right)^{1/2} < V < \frac{\pi}{2} + \frac{1}{2}\tan^{-1}\left(\frac{n_3^2 - n_1^2}{n_2^2 - n_3^2}\right) \qquad (5.46)$$

NB. For a symmetric waveguide, (5.46) reveals that the TE_0 mode exists at all frequencies.

Consider the low and high frequency limits as shown in the above figure...

In the low frequency limit:

↳ modes approach cutoff frequencies

(5.30) $\Rightarrow \beta = n_3 \omega / c$

same propagation constant as the undercladding

$\Rightarrow$ leaky mode

radiation mode, if waveguide is symmetric

"unbound"

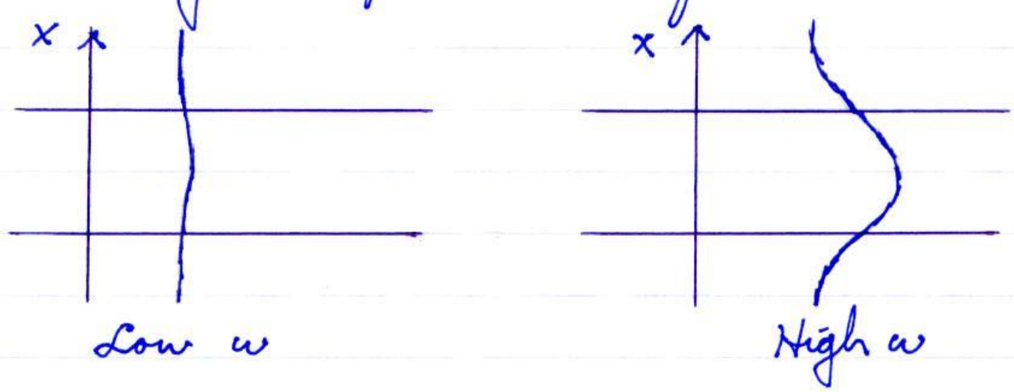
In the high frequency limit:

$\hookrightarrow \beta \to n_2 \omega / c$ ...from (5.28)

"asymptotically"

same propagation constant as core

$\Rightarrow$ tightest possible confinement



Low ω High ω

Effective Index

We may define the effective index for a guided mode as

$$n_{eff} = \beta / k_0 \tag{5.47}$$

From (5.30) & (5.28):

$$n_3 \leq n_{eff} < n_2 \tag{5.48}$$

Let us introduce a new parameter:

$$b = \frac{n_{eff}^2 - n_3^2}{n_2^2 - n_3^2} \tag{5.49}$$

normalized propagation constant

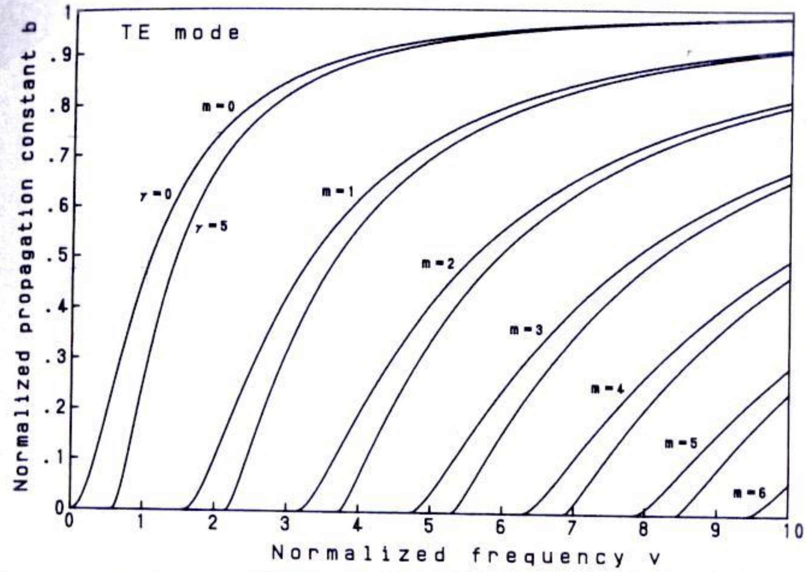
(5.48) $\Rightarrow$ $\quad 0 \leq b < 1$

$b = 0$ at cutoff

NB. The index asymmetry may also be quantified:

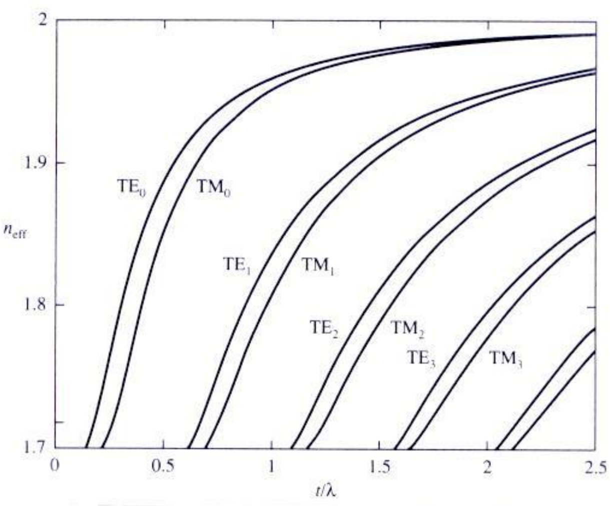
$\gamma = 0$, symmetric

$$\gamma = \frac{n_3^2 - n_1^2}{n_2^2 - n_3^2} \tag{5.50}$$



K. Okamoto, Fundamentals of Optical Waveguides, 2[ND] ed., p. 23.

Similar dispersion relations and mode cutoff conditions may be found for TM modes. A comparison of TE & TM is shown below, for $n_1 = 1.0$, $n_2 = 2.0$, $n_3 = 1.7$



Yariv & Yeh, p. 123

NB. TE modes are more tightly confined than TM modes

Mode Orthogonality

Consider two linearly independent solutions...

$$[\vec{\mathcal{E}}_m(\vec{r},t), \vec{\mathcal{H}}_m(\vec{r},t)] \ \& \ [\vec{\mathcal{E}}_n(\vec{r},t), \vec{\mathcal{H}}_n(\vec{r},t)]$$

... to Maxwell's curl equations (1.3) & (1.4), where $\vec{\mathcal{E}}$ & $\vec{\mathcal{H}}$ are given by (5.3) & (5.4) as...

$$\vec{\mathcal{E}}(\vec{r},t) = \vec{E}(x,y)\, e^{i(\omega t - \beta z)}$$

$$\vec{\mathcal{H}}(\vec{r},t) = \vec{H}(x,y)\, e^{i(\omega t - \beta z)}$$

Since dielectric waveguides have no charge or current sources, then

$$\vec{\nabla} \cdot \left(\vec{\mathcal{E}}_m \times \vec{\mathcal{H}}_n - \vec{\mathcal{E}}_n \times \vec{\mathcal{H}}_m \right) = 0 \qquad (5.51)$$

$\hookrightarrow$ Lorentz reciprocity theorem

assuming $[\vec{\mathcal{E}}_m, \vec{\mathcal{H}}_m] \rightarrow \beta_m$

$[\vec{\mathcal{E}}_n, \vec{\mathcal{H}}_n] \rightarrow \beta_n$

and $\vec{\nabla} \rightarrow \vec{\nabla}_t + \hat{z}\, \partial / \partial z$, then...

... (5.51) reduces to

$$\vec{\nabla}_t \cdot (\vec{\mathcal{E}}_m \times \vec{\mathcal{H}}_n - \vec{\mathcal{E}}_n \times \vec{\mathcal{H}}_m) \qquad (5.52)$$

$$- i(\beta_m + \beta_n)\hat{z} \cdot (\vec{\mathcal{E}}_m \times \vec{\mathcal{H}}_n - \vec{\mathcal{E}}_n \times \vec{\mathcal{H}}_m) = 0$$

Applying the divergence theorem to (5.52) yields

$$\int_S \vec{\nabla}_t \cdot (\vec{\mathcal{E}}_m \times \vec{\mathcal{H}}_n - \vec{\mathcal{E}}_n \times \vec{\mathcal{H}}_m)\, dS = \oint_C (\vec{\mathcal{E}}_m \times \vec{\mathcal{H}}_n - \vec{\mathcal{E}}_n \times \vec{\mathcal{H}}_m) \cdot \hat{n}\, dl \qquad (5.53)$$

$$= i(\beta_m + \beta_n) \int_S (\vec{\mathcal{E}}_m \times \vec{\mathcal{H}}_n - \vec{\mathcal{E}}_n \times \vec{\mathcal{H}}_m) \cdot \hat{z}\, dS$$

Since S is an arbitrary transverse waveguide cross section, whose boundary is C, the over the entire mode $(S \to \infty)$, the contour integral vanishes since the field vanishes.
Hence, (5.53) yields

$$(\beta_m + \beta_n) \int_S^{\infty} (\vec{\mathcal{E}}_m \times \vec{\mathcal{H}}_n - \vec{\mathcal{E}}_n \times \vec{\mathcal{H}}_m) \cdot \hat{z}\, dS = 0 \qquad (5.54)$$

(5.3) & (5.4) into (5.54) yield...

$$(\beta_m + \beta_n) \int_S^{\infty} (\vec{E}_m \times \vec{H}_n - \vec{E}_n \times \vec{H}_m) \cdot \hat{z} \, dS = 0 \qquad (5.55)$$

now, exploit symmetry:

↳ mirror transformations about $z = 0$ plane (ie. $z \to -z$) are also solutions ... with the same β.

The mode transforms as $\vec{\mathcal{E}} \to \vec{\mathcal{E}}'$, $\vec{\mathcal{H}} \to \vec{\mathcal{H}}'$, where

$$\vec{\mathcal{E}}' = \begin{cases} \vec{\mathcal{E}}_t' = \vec{E}_t \, e^{i(\omega t + \beta z)} \\ \vec{\mathcal{E}}_z' = -\vec{E}_z \, e^{i(\omega t + \beta z)} \end{cases} \qquad (5.56)$$

zero for TE

$$\vec{\mathcal{H}}' = \begin{cases} \vec{\mathcal{H}}_t' = -\vec{H}_t \, e^{i(\omega t + \beta z)} \\ \vec{\mathcal{H}}_z' = +\vec{H}_z \, e^{i(\omega t + \beta z)} \end{cases}$$

zero for TM

If we employ (5.56), for $[\vec{\mathcal{E}}_n, \vec{\mathcal{H}}_n]$, in (5.54), then (5.55) becomes

$$(\beta_m - \beta_n) \int_S^{\infty} (-\vec{E}_m \times \vec{H}_n - \vec{E}_n \times \vec{H}_m) \cdot \hat{z} \, dS = 0 \qquad (5.57)$$

Addition and subtraction of (5.55) & (5.57) yields

$$\int_S^\infty (\vec{E}_m \times \vec{H}_n) \cdot \hat{z} \, dS = \int_S^\infty (\vec{E}_n \times \vec{H}_m) \cdot \hat{z} \, dS = 0 \qquad (5.58)$$

general orthogonality condition

In lossless waveguides, the derivation may proceed from (5.54) by...

$$[\vec{\mathcal{E}}_n, \vec{H}_n] \rightarrow [\vec{\mathcal{E}}_n^*, \vec{H}_n^*]$$

This yields

$$\int_S^\infty (\vec{\mathcal{E}}_m \times \vec{H}_n^*) \cdot \hat{z} \, dS \qquad (5.59)$$

$$= \int_S^\infty (\vec{\mathcal{E}}_n \times \vec{H}_m) \cdot \hat{z} \, dS = 0$$

In comparison with the time-averaged Poynting vector (1.68), (5.59) implies that the power flow in lossless waveguides is the sum of the power in each mode.