Ch. 4 Optical Beams

geometrical optics

$\hookrightarrow$ point focus of rays, no diffraction…

4.1 The Wave Equation in Quadratic Index Media

Recall (1.77)

$$\nabla^2\mathbf{E} - \mu\varepsilon\frac{\delta^2\mathbf{E}}{\delta t^2} = -\nabla\left(\varepsilon^{-1}\mathbf{E}\cdot\nabla\varepsilon\right)$$

If the medium is sufficiently homogenous that

$$\frac{\nabla\varepsilon}{\varepsilon} \ll 1 \quad \text{over } \lambda$$

…then we recover

$$\nabla^2\mathbf{E} + k^2(\mathbf{r})\mathbf{E} = 0 \qquad (4.1)$$

$\hookleftarrow$ Helmholtz equation

where, in general, from (1.48) & (2.63),

$$k^2(\mathbf{r}) = \omega^2\mu\varepsilon_e(\mathbf{r}) = \omega^2\mu\varepsilon(\mathbf{r})\left[1 + \frac{\sigma(\mathbf{r})}{j\omega\varepsilon(\mathbf{r})}\right] \qquad (4.2)$$

What are the solutions of (4.1)?

$\hookrightarrow$ One solution…

"*Gaussian Beam*"

i.e. Planes perpendicular to propagation direction have a Gaussian intensity distribution

We take a general approach to show this.

Assume a lossless medium with lenslike symmetry about the propagation direction z. Let $\rho^2 = x^2 + y^2$ be the radial coordinate in the transverse plane. Then

$$n^2(\mathbf{r}) = \frac{\varepsilon(\mathbf{r})}{\varepsilon_0} = n_0^2\left[1 - g^2\rho^2\right] \qquad (4.3)$$

where: n_0 – index on symmetry axis

g – a constant such that $g^2\rho^2 << 1$

$$\nabla^2 = \nabla_t^2 + \frac{\partial^2}{\partial z^2} \qquad (4.4)$$

Cylindrical symmetry:

$$\nabla_t^2 \to \frac{\partial^2}{\partial \rho^2} + \frac{1}{\rho}\frac{\partial}{\partial \rho} \qquad (4.5)$$

Since $g^2\rho^2 << 1$, energy flow is primarily along z-axis

→ "*nearly*" plane wave

→ need only consider a single transverse field component…

$$\mathbf{E}(\rho, z) = \hat{\mathbf{a}}\psi(\rho, z)e^{-ik_0 z} \qquad (4.6)$$

$$\text{where : } \quad k_0 = \omega n_0 / c \qquad (4.7)$$

Expressing (4.1) with (4.4), using (4.3) & (4.7):

$$\nabla_t^2\mathbf{E} + \frac{\partial^2\mathbf{E}}{\partial z^2} + k_0^2\left(1 - g^2\rho^2\right)\mathbf{E} = 0 \qquad (4.8)$$

(4.6) → (4.8) yields…

$$\nabla_t^2\psi - i2k_0\frac{2\psi}{\delta z} - k_0^2 g^2\rho^2\psi = 0 \qquad (4.9)$$

…where we have used the "*slowly varying envelope approximation*"

$$\frac{\partial^2 \psi}{\partial z^2} \ll \frac{\partial \psi}{\partial z}$$

to neglect the $\partial^2 \psi / \partial z^2$ term.

Let us assume that (4.9) has a solution of the form

$$\psi = \psi_0 \exp\left\{ -i \left[\underbrace{P(z)}_{\text{phase}} + \frac{k_0 \rho^2}{2 \underbrace{q(z)}_{\text{radius of curvature (complex)}}} \right] \right\} \qquad (4.10)$$

What are the constraints on $P(z)$ & $q(z)$?

(4.10) → (4.9), using (4.4) & (4.5), and first considering each term independently…

$$\frac{\partial \psi}{\partial \rho} = \frac{-ik_0 \rho}{q(z)} \psi \qquad \frac{\partial^2 \psi}{\partial \rho^2} = \frac{-ik_0}{q(z)} \left\{ 1 - \frac{ik_0 \rho^2}{q(z)} \right\} \psi$$

$$\frac{\partial \psi}{\partial z} = -i \left\{ \frac{\partial P}{\partial z} + \frac{k_0 \rho^2}{2} \frac{\partial}{\partial z} \left(\frac{1}{q} \right) \right\} \psi$$

…yields

$$k_0 \left\{ \frac{\partial}{\partial z} \left(\frac{1}{q(z)} \right) + \frac{1}{q^2(z)} + g^2 \right\} \rho^2 + 2 \left\{ \frac{\partial P}{\partial z} + \frac{i}{q(z)} \right\} \rho^0 = 0 \qquad (4.11)$$

To satisfy (4.11) for all ρ, each coefficient of ρ must vanish. Hence

$$\frac{\partial}{\partial z} \left(\frac{1}{q(z)} \right) + \frac{1}{q^2(z)} + g^2 = 0 \qquad (4.12)$$

$$\frac{\partial P}{\partial z} = -\frac{i}{q(z)} \qquad (4.13)$$

4.2 Gaussian Beams in Homogenous Media

For Gaussian beams, g = 0. (4.12) then becomes

$$\frac{\partial}{\partial z}\left(\frac{1}{q(z)}\right)+\frac{1}{q^2(z)}=0 \qquad (4.14)$$

To solve (4.14), let

$$\frac{1}{q(z)}=\frac{1}{u}\frac{\partial u}{\partial z} \qquad (4.15)$$

(4.15) → (4.14) yields

$$\frac{\partial}{\partial z}\left(\frac{1}{u}\frac{\partial u}{\partial z}\right)+\frac{1}{q^2(z)}=0$$

$$-\frac{1}{u^2}\left(\frac{\partial u}{\partial z}\right)^2+\frac{1}{u}\frac{\partial^2 u}{\partial z^2}+\frac{1}{q^2(z)}=0$$

(4.15) then requires

$$\frac{\partial^2 u}{\partial z^2}=0 \qquad (4.16)$$

(4.16) has solutions of the form

$$u(z)=az+b \qquad (4.17)$$

(4.17) → (4.15) yields

$$q(z)=z+q_0 \qquad (4.18)$$

q_0 = b/a, a constant

(4.18) into (4.13) yields

$$\frac{\partial P}{\partial z}=-\frac{i}{z+q_0} \qquad (4.19)$$

Integrating (4.19) yields

$$P(z)=-i\int\frac{dz}{z+q_0}=-i\ln\left(1+z/q_0\right) \qquad (4.20)$$

NB. Here, the arbitrary constant has been set to zero since it merely introduces a phase shift.

We can now construct (4.10). Consider its total phase, using (4.18) & (4.20)

$$-i\left[P(z)+\frac{k_0\rho^2}{2q(z)}\right]=-\ln\left(1+\frac{z}{q_0}\right)-\frac{ik_0\rho^2}{2(z+q_0)} \qquad (4.21)$$

(4.21) into (4.10) yields

$$\psi(\rho,z)=\psi_0\exp\left[-\ln\left(1+z/q_0\right)\right]\exp\left[\frac{-ik_0\rho^2}{2(z+q_0)}\right]$$

$$=\psi_0\left(\frac{q_0}{z+q_0}\right)\exp\left[\frac{-ik_0\rho^2}{2(z+q_0)}\right] \qquad (4.22)$$

Applying… $\psi\to 0$ as $\rho\to\infty$

Sommerfeld radiation condition

…then q_0 must be positive pure imaginary, else (4.22) will diverge

$$\Rightarrow q_0=iz_0 \quad , z_0>0 \qquad (4.23)$$

We can define a constant ω_0 such that

$$z_0=\frac{1}{2}k_0\omega_0=\frac{\pi n_0\omega_0^2}{\lambda} \qquad (4.24)$$

Physically, what is ω_0? Consider (4.22) at $z = 0$:

$$\psi(\rho,0)=\psi_0\exp\left(\frac{-ik_0\rho^2}{2q_0}\right)=\psi_0\exp\left(\frac{-\rho^2}{\omega_0^2}\right) \qquad (4.25)$$

using (4.23) & (4.24)

Gaussian distribution with a spot size of ω_0

For $z\neq 0$, from (4.22):

$$\frac{-ik_0\rho^2}{2(z+q_0)}=\frac{-\rho^2}{\omega_0^2-i\,2z/k_0}=\frac{-\rho^2\left(1+i\,z/z_0\right)}{\omega_0^2\left(1+z^2/z_0^2\right)} \qquad (4.26)$$

$$\frac{q_0}{z+q_0} = \frac{1}{1-i\, z/z_0} = \frac{1}{\sqrt{1+z^2/z_0^2}} \angle \tan^{-1}\left(z/z_0\right) \qquad (4.27)$$

(4.26) & (4.27) → (4.22) yields

$$\psi(\rho,z) = \frac{\psi_0 e^{i\tan^{-1}\left(z/z_0\right)}}{\sqrt{1+z^2/z_0^2}} \exp\left\{\frac{-\rho^2\left(1+i\, z/z_0\right)}{\omega_0^2\left(1+z^2/z_0^2\right)}\right\} \qquad (4.28)$$

Define beam parameters…

$$\omega^2(z) = \omega_0^2\left\{1+z^2/z_0^2\right\} \qquad (4.29)$$

$$R(z) = z\left\{1+z_0^2/z^2\right\} \qquad (4.30)$$

$$\eta(z) = \tan^{-1}(z/z_0) \qquad (4.31)$$

where, from (4.24)

$$\omega_0 = \sqrt{\frac{\lambda z_0}{\pi n_0}} \qquad (4.32)$$

…then, in (4.28)

$$\frac{-\rho^2\left(1+i\, z/z_0\right)}{\omega_0^2\left(1+z^2/z_0^2\right)} = \frac{-\rho^2}{\omega^2(z)} - \frac{iz_0\rho^2}{\omega_0^2 z\left(1+z^2/z_0^2\right)} = \frac{-\rho^2}{\omega^2(z)} - i\frac{k_0\rho^2}{2R(z)} \qquad (4.33)$$

…using (4.24), (4.29) & (4.30).

(4.28) into (4.6), using (4.33) yields

$$E(\rho,z) = E_0\left(\omega_0/\omega(z)\right) e^{-\rho^2/\omega^2(z)} e^{-i\varphi(\rho,z)} \qquad (4.34)$$

where

$$\varphi(\rho,z) = k_0\left\{z+\rho^2/2R(z)\right\} - \eta(z) \qquad (4.35)$$

phase of the Gaussian beam

4.3 Properties of the Gaussian Beam

- **Intensity**: $I(\rho,z)=\left|E(\rho,z)\right|^2$ (4.36)

(4.34) → (4.36) yields

$$I(\rho,z)=I_0\left(\frac{\omega_0}{\omega(z)}\right)^2\exp\left\{\frac{-2\rho^2}{\omega^2(z)}\right\} \qquad (4.37)$$

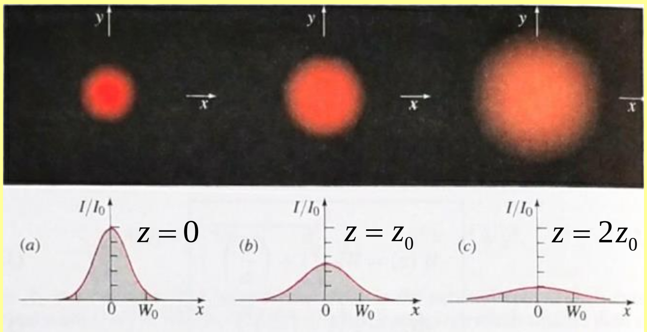

Fig 4.1 Saleh & Teich, 2nd ed. p. 78

NB. $I(\rho, z)$ is Gaussian at ANY z, the beam width $\omega(z)$ increasing with z as per (4.29)

On the beam axis, (4.29) → (4.37) shows

$$I(0,z)=\frac{I_0}{1+z^2/z_0^2} \qquad (4.38)$$

NB. For $z >> z_0$, $I(0, z) \approx I_0 z_0^2/z^2$ → obeys inverse-square law.

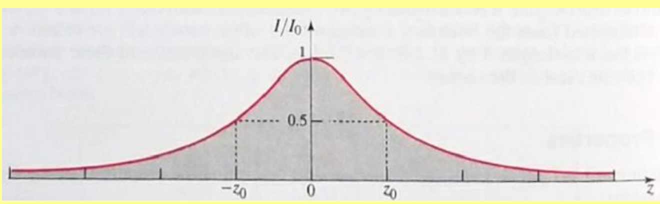

Fig 4.2 Saleh & Teich, 2nd ed. p. 78

- **Power**: $P=\int_0^\infty I(\rho,z)2\pi\rho d\rho$ (4.39)

optical intensity over any transverse plane z

(4.37) → (4.39) yields

$$P = 2\pi I_0 \left(\frac{\omega_0}{\omega(z)}\right)\int_0^\infty \rho \exp\left(\frac{-2\rho^2}{\omega^2(z)}\right) d\rho = \frac{1}{2} I_0 \left(\pi\omega_0^2\right) \qquad (4.40)$$

Beam area

…independent of z, as expected.

- **Beam Width**: From (4.37), we see that the intensity drops by e^{-2} at $\rho = \omega(z)$, where $\omega(z)$ is given by (4.29)

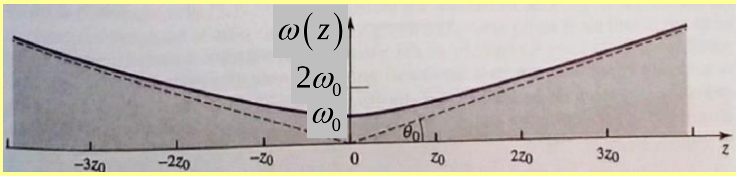

Fig. 4.3 Saleh & Teich, 2nd ed, p. 79

NB. $\omega_0 = \omega(z=0)$ ← the minimum

Waist radius

- **Beam Divergence:** The asymptotes of $\omega(z)$ define θ_0.

For $z >> z_0$, (4.29) may be approximated as

$$\omega(z) \approx \left(\frac{\omega_0}{z_0}\right) z \equiv \theta_0 z \qquad (4.41)$$

Using the beam parameter (4.32) → (4.41), the *half angle* is

$$\theta_0 = \frac{\omega_0}{z_0} = \frac{\omega_0 \lambda}{\pi n_0 \omega_0^2} = \frac{\lambda}{\pi n \omega_0} \qquad (4.42)$$

and the divergence angle then is

$$2\theta_0 = \frac{2\lambda}{\pi n \omega_0} \qquad (4.43)$$

NB. A well-collimated beam must have a large diameter.

- **Depth of Focus**: A *Confocal parameter*

  The best focus is at z = 0. Define the confocal parameter as that distance Δz within which the beam area is no greater than twice its minimum.

  Applying this definition to (4.29) & (4.32)…

$$2\left(\pi\omega_0^2\right) = \pi\omega^2(z)$$

$$2 = 1 + z^2/z_0^2$$

$$\Rightarrow \Delta z = z_0 - (-z_0) = 2z_0 = \frac{2\pi n\omega_0^2}{\lambda} \qquad (4.44)$$

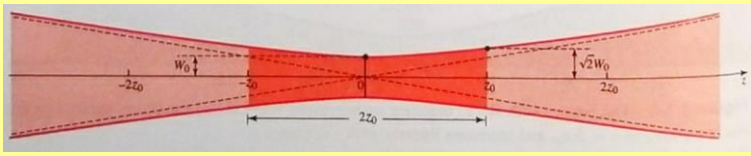

Fig. 4.4 Saleh & Teich, 2nd ed., p. 81

- **Phase**: The phase of the beam is given by (4.35) as

$$\varphi(\rho, z) = k\left\{z + \frac{\rho^2}{2R(z)}\right\} - \eta(z)$$

dropping the subscript on k

On the beam axis, this reduces to

$$\varphi(0, z) = kz - \eta(z) \qquad (4.45)$$

phase of a plane wave

phase retardation

Gouy phase shift

From (4.31):

$$-\frac{\pi}{2} = \eta(-\infty) < \eta(z) < \eta(+\infty) = \frac{\pi}{2}$$

→ accumulated Gouy phase shift = π

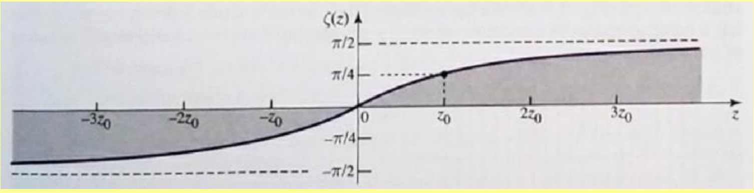

Fig. 4.5 Saleh & Teich, 2nd ed., p. 81

- **Wavefronts**: In (4.35), the radial component…

$$\rho^2 / 2R(z)$$

… is responsible for wavefront bending

gives the phase deviation within the transverse plane from the axial value.

Surfaces of constant phase…

$$\varphi(\rho, z) = C \qquad (4.46)$$

some arbitrary constant

(4.35) → (4.46) yields

$$k\left\{z + \frac{\rho^2}{2R(z)}\right\} - \eta(z) = C$$

$$\Rightarrow \frac{\rho^2}{2R(z)} = \frac{\lambda(C + \eta(z))}{2\pi n} - z \qquad (4.47)$$

NB. $R(z)$ & $\eta(z)$ are slowly varying functions of z on any wavefront for $\rho < \omega(z)$

Treating R and η as constants, (4.47) then describes a paraboloidal surface

→ $R(z)$ – *radius of curvature*

given by (4.30) and plotted below, along with the corresponding wavefronts.

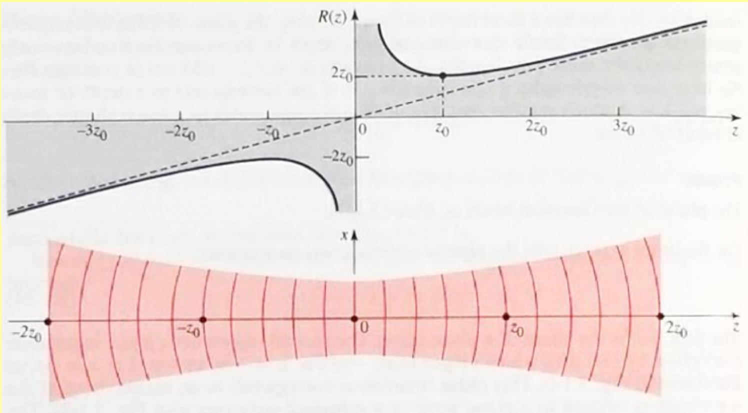

Fig. 4.6 Saleh & Teich, 2nd ed, p. 82

NB. At $z = 0$, $R(z) \to \infty$

$\to$ planar wavefronts

At $z = z_0$, $R(z_0) = 2z_0$ is a minimum

- **Beam Quality**: a measure of the deviation of an optical beam from Gaussian form

$\to$ given by M^2 — the ratio of the beam "waist diameter – divergence product" relative to a Gaussian beam

For a Gaussian beam in free space

$$2\omega_0 2\theta_0 = 2\omega_0 \left(\frac{2\lambda}{\pi \omega_0} \right), \text{ using (4.43)}$$

$$= \frac{4\lambda}{\pi} \qquad (4.48)$$

Therefore:

$$M^2 = \frac{2\omega_m 2\theta_m}{4\lambda/\pi} \geq 1 \qquad (4.49)$$

NB. For an optical beam with the same waist diameter as a Gaussian...

$M^2 \rightarrow$ a beam divergence angle M^2 greater than the minimum.

Let us return to (4.34) and more carefully consider this solution and the terms it contains.

(4.18) and (4.23) give

$$\frac{1}{q(z)} = \frac{1}{z + iz_0} \qquad (4.50)$$

(4.50) may be rewritten using the beam parameter equations (4.29), (4.30) and (4.32) as

$$\frac{1}{q(z)} = \frac{1}{R(z)} - \frac{i\lambda}{\pi n \omega^2(z)} \qquad (4.51)$$

complex radius of curvature of the Gaussian beam

aka:

Gaussian Beam Parameter

radius of curvature (of the very nearly spherical beam) at z

Then from (4.34), the Gaussian beam solution is…

$$E(\rho, z) = E_0 \left[\frac{\omega_0}{\omega(z)}\right] \exp\left[\frac{-\rho^2}{\omega^2(z)}\right] \exp\left\{-i\left[kz - \eta(z)\right] - \frac{ik\rho^2}{2R(z)}\right\} \qquad (4.52)$$

NB. It is the *fundamental* solution because our insistence on radial symmetry – see (4.3) – has excluded the higher order modes with their azimuthal symmetry.

Does this solution make sense? Consider a spherical wave emitted by a point source:

$$E \sim \frac{e^{-ikr}}{r} \qquad (4.53)$$

In the paraxial approximation…

$$r = \sqrt{x^2 + y^2 + z^2} = z\sqrt{1 + \rho^2 / z^2}$$

$$\simeq z\left(1 + \rho^2 / 2z^2\right) \equiv z + \rho^2 / 2R^2$$

Thus, (4.53) becomes

$$E \sim R^{-1} \exp\left(-ikz - \frac{ik\rho^2}{2R}\right) \qquad (4.54)$$

Comparing (4.34) and (4.52) with (4.54) confirms our understanding of R and q as radii of curvature.