

Ch. 3 Rays

Geometrical optics limit...

- beam diameter \gg wavelength
diffraction neglected
- obey Fermat's principle of "least time"

- rays travel in straight lines through homogenous media

* We want to trace ray trajectories through optical elements...

e.g., lenses, interfaces, mirrors...

3.1 Ray Matrices

Some definitions...

- Paraxial ray: rays with negligible angular deviations from the optical axis (z -axis) $\rightarrow \sin \theta \approx \tan \theta \approx \theta$

Lecture 8: ELG6372 – Principles of Photonics, H. Schriemer

1

- Thin lens: focal length f , thickness irrelevant, cylindrical symmetry

Consider a paraxial ray traversing a thin lens...

$\rightarrow \rho$: the distance from the optical axis

...and the class of rays described by $\rho(z)$ as:

Meridional rays – trajectories confined in a plane through the optical axis

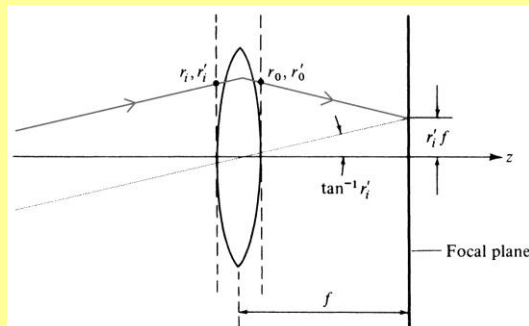


Fig. 3.1: Deflection of a ray by a thin lens. From Yariv & Yeh, 6th ed., p. 67. NB. Their r is our ρ .

NB. $f > 0$, convergent lens;
 $f < 0$, divergent lens

Lecture 8: ELG6372 – Principles of Photonics, H. Schriemer

2

*For a thin lens:

$$\rho_{\text{out}} = \rho_{\text{in}} \quad (3.1)$$

└─ ray output & input positions

$$\rho'_{\text{out}} = \rho'_{\text{in}} - \rho_{\text{in}} / f \quad (3.2)$$

└─ ray output & input slopes

The matrix representation of (3.1) & (3.2) is

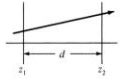
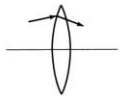
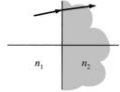
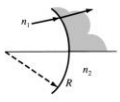
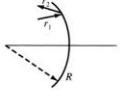
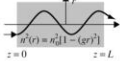
$$\begin{bmatrix} \rho_{\text{out}} \\ \rho'_{\text{out}} \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ -1/f & 1 \end{bmatrix} \begin{bmatrix} \rho_{\text{in}} \\ \rho'_{\text{in}} \end{bmatrix} \quad (3.3)$$

└─ input ray vector

output ─┬─
ray vector

└─ ray matrix

└─ represents the effect of the optical element

(1) Straight Section: Length d		$\begin{bmatrix} 1 & d \\ 0 & 1 \end{bmatrix}$
(2) Thin Lens: Focal length f ($f > 0$, converging; $f < 0$, diverging)		$\begin{bmatrix} 1 & 0 \\ -1/f & 1 \end{bmatrix}$
(3) Dielectric Interface: Refractive indices n_1, n_2		$\begin{bmatrix} 1 & 0 \\ 0 & \frac{n_1}{n_2} \end{bmatrix}$
(4) Spherical Dielectric Interface: Interface: Radius R		$\begin{bmatrix} 1 & 0 \\ \frac{n_2 - n_1}{n_2 R} & \frac{n_1}{n_2} \end{bmatrix}$
(5) Spherical Mirror: Radius of curvature R		$\begin{bmatrix} 1 & 0 \\ -2/R & 1 \end{bmatrix}$
(6) A medium with a quadratic index profile		$\begin{bmatrix} \cos(gz) & \sin(gz)/g \\ -g \sin(gz) & \cos(gz) \end{bmatrix}$

e.g., For propagation through a homogeneous medium of length d ...

$$\begin{bmatrix} \rho_{\text{out}} \\ \rho'_{\text{out}} \end{bmatrix} = \begin{bmatrix} 1 & d \\ 0 & 1 \end{bmatrix} \begin{bmatrix} \rho_{\text{in}} \\ \rho'_{\text{in}} \end{bmatrix} \quad (3.4)$$

Table 3.1: Ray matrices for some optical elements and media. From Yariv & Yeh, 6th ed., p. 68. NB. Their r is our ρ .

Propagation through a periodic lens system is equivalent to Gaussian beam propagation inside an optical resonator whose mirrors are separated by d and have radii of curvature $R_1 = 2f_1$ and $R_2 = 2f_2$

Repeated units of lenses (see Fig. 3.2) of focal lengths f_1 & f_2 separated by distance d

→ unit cell between planes s & $s + 1$

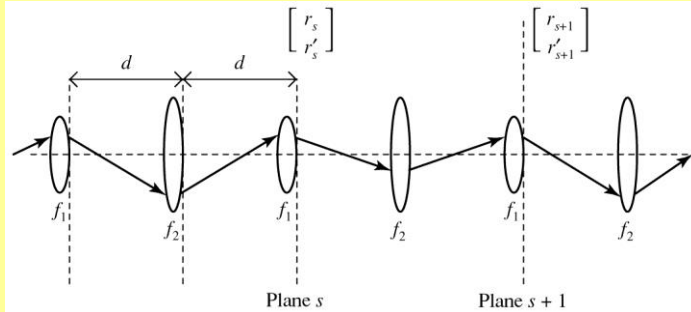


Fig. 3.2: The Periodic Lens Waveguide. From Yariv & Yeh, 6th ed., p. 69. NB. Their r is our ρ .

The matrix representation from input to output of this unit cell is

$$\begin{bmatrix} \rho_{s+1} \\ \rho'_{s+1} \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ -1/f_1 & 1 \end{bmatrix} \begin{bmatrix} 1 & d \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ -1/f_2 & 1 \end{bmatrix} \begin{bmatrix} 1 & d \\ 0 & 1 \end{bmatrix} \begin{bmatrix} \rho_s \\ \rho'_s \end{bmatrix} \quad (3.5)$$

(3.5) works out to

$$\begin{bmatrix} \rho_{s+1} \\ \rho'_{s+1} \end{bmatrix} = \begin{bmatrix} A & B \\ C & D \end{bmatrix} \begin{bmatrix} \rho_s \\ \rho'_s \end{bmatrix} \quad (3.6)$$

where the matrix elements are

$$A = 1 - d/f_2 \quad (3.7a)$$

$$B = d \left(2 - d/f_2 \right) \quad (3.7b)$$

$$C = -\left[\frac{1}{f_1} + \frac{1}{f_2}\left(1 - \frac{d}{f_1}\right)\right] \quad (3.7c)$$

$$D = -\left[\frac{d}{f_1} - \left(1 - \frac{d}{f_1}\right)\left(1 - \frac{d}{f_2}\right)\right] \quad (3.7d)$$

NB. This ray matrix is a unimodular matrix...

$$AD - BC = 1 \quad (3.8)$$

Applying (3.6) to the propagation through N unit cells...

$$\begin{bmatrix} \rho_N \\ \rho'_N \end{bmatrix} = \begin{bmatrix} A & B \\ C & D \end{bmatrix}^N \begin{bmatrix} \rho_0 \\ \rho'_0 \end{bmatrix} \quad (3.9)$$

The Chebyshev identity can be used to find the N^{th} power of the matrix:

$$\begin{bmatrix} A & B \\ C & D \end{bmatrix}^N = \begin{bmatrix} AU_{N-1} - U_{N-2} & BU_{N-1} \\ CU_{N-1} & DU_{N-1} - U_{N-2} \end{bmatrix} \quad (3.10)$$

where
$$U_N = \frac{\sin[(N+1)\theta]}{\sin\theta} \quad (3.11)$$

with
$$\theta = \cos^{-1}\left[\frac{1}{2}(A+D)\right] \quad (3.12)$$

NB. Chebyshev's identity may be proved by induction.

Now, consider the confinement of the ray by the lens system. For finite input parameters ρ_0 & ρ'_0 (at plane 0), the parameters ρ_N & ρ'_N must remain finite as $N \rightarrow \infty$. Inspection of (3.12) reveals that θ must be real. Therefore

$$\left|\frac{A+D}{2}\right| \leq 1 \quad (3.13)$$

(3.7a) & (3.7d) \rightarrow (3.13) yields the confinement condition

$$0 \leq \left(1 - \frac{d}{2f_1}\right) \left(1 - \frac{d}{2f_2}\right) \leq 1 \quad (3.14)$$

What happens if the ray is unconfined? Since θ is then imaginary, $\theta = iq$ (where q is real). In (3.11), the sine functions become hyperbolic, whence

$$U_N = \frac{\sinh[(N+1)q]}{\sinh q} \quad (3.15)$$

$\rightarrow \rho_N$ & ρ'_N grow exponentially with N

*Special Case: The identical lens waveguide, $f_1 = f_2 \equiv f$

\hookrightarrow The confinement condition (3.14) becomes...

$$\left(1 - \frac{d}{2f}\right)^2 \leq 1 \quad (3.16)$$

$$\Rightarrow 0 \leq 1 - \frac{d}{4f} \leq 1$$

The 1st inequality gives $d \leq 4f$, while the 2nd requires $f > 0$ (i.e., only converging lenses) since $d \geq 0$. The confinement condition may be given as

$$0 \leq d \leq 4f \quad (3.17)$$

We can solve the special case directly. Going from plane s to $(s+1)$, the ray parameters are

$$\begin{bmatrix} \rho_{s+1} \\ \rho'_{s+1} \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ -1/f_1 & 1 \end{bmatrix} \begin{bmatrix} 1 & d \\ 0 & 1 \end{bmatrix} \begin{bmatrix} \rho_s \\ \rho'_s \end{bmatrix} = \begin{bmatrix} A & B \\ C & D \end{bmatrix} \begin{bmatrix} \rho_s \\ \rho'_s \end{bmatrix} \quad (3.18)$$

\uparrow propagator matrix
 \uparrow lens matrix

NB. Since $f_1 = f_2 \equiv f$, the unit cell for the special case has only ONE lens in it, unlike the general case - see Fig 3.2

The matrix multiplication yields

$$\begin{bmatrix} A & B \\ C & D \end{bmatrix} = \begin{bmatrix} 1 & d \\ -1/f & 1-d/f \end{bmatrix} \quad (3.19)$$

From (3.13), the confinement condition becomes...

$$\left| 1 - \frac{d}{2f} \right| \leq 1$$

Or:
$$-1 \leq 1 - \frac{d}{2f} \leq 1 \quad (3.20)$$

...which is equivalent to (3.16) & (3.17).

*As shown in Problem 2.2, text, the ray location (or beam radius) at the m^{th} lens (where m denotes the plane immediately to the right of the m^{th} lens) can be written as

$$\rho_m = \rho_{\max} \sin(m\theta + \alpha) \quad (3.21)$$

From (3.19) \rightarrow (3.12) we note...

$$\cos \theta = 1 - \frac{d}{2f} \quad (3.22)$$

...and ρ_{\max} & α may be found as

$$\rho_{\max}^2 = \frac{4f \left[\rho_0^2 + d\rho_0\rho_0' + df(\rho_0')^2 \right]}{4f - d} \quad (3.23)$$

$$\tan \alpha = \frac{\sqrt{4f/d - 1}}{1 + 2f\rho_0'/\rho_0} \quad (3.24)$$

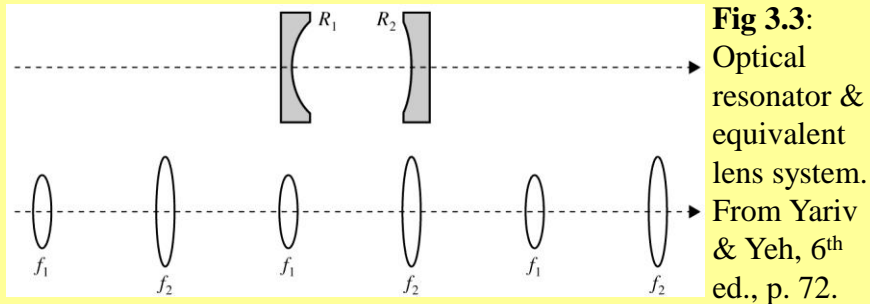
Applications: Rays in Optical Resonators

Two curved mirrors ← form an optical resonator

└ rays bounce between

NB. Reflection at a mirror (R_1 – radius of curvature) is equivalent to traversing a lens (f , focal length) if

$$f = \frac{1}{2} R \quad (3.25)$$



For an optical resonator with two concave mirrors separated by d , having radii of curvature R_1 & R_2 , the confinement condition (3.14) becomes, using (3.25)

$$0 \leq \left(1 - \frac{d}{R_1}\right) \left(1 - \frac{d}{R_2}\right) \leq 1 \quad (3.26)$$

3.2 Skew Rays & Reentrant Rays

Skew Rays – the general case

└ Ray propagates with a 3D trajectory, neither crossing the optical axis nor being parallel to it

→ x & y are independent coordinates

Description of ray propagation requires...

$$x(z), x'(z) \text{ \& \ } y(z), y'(z)$$

e.g. The Lens Waveguide

→ ray parameters obey the same ray matrix, therefore...

$$\begin{bmatrix} x_N \\ x'_N \end{bmatrix} = \begin{bmatrix} A & B \\ C & D \end{bmatrix}^N \begin{bmatrix} x_0 \\ x'_0 \end{bmatrix} \quad (3.27)$$

$$\begin{bmatrix} y_N \\ y'_N \end{bmatrix} = \begin{bmatrix} A & B \\ C & D \end{bmatrix}^N \begin{bmatrix} y_0 \\ y'_0 \end{bmatrix} \quad (3.28)$$

...as per (3.9)

As per (3.21), the ray position at the N^{th} plane is

$$x_N = x_{\max} \sin(N\theta + \alpha_x) \quad (3.29)$$

$$y_N = y_{\max} \sin(N\theta + \alpha_y) \quad (3.30)$$

→ locus of points (x_N, y_N) confined to an ellipse bounded by a rectangle of $2x_{\max}$ by $2y_{\max}$ (for θ real) – cf. (1.172)

NB. For $\alpha_y - \alpha_x = 0, \pi$

↳ ray trajectory confined to the meridional plane

Reentrant Rays

(3.29) & (3.30) → ray positions have a sinusoidal dependence on $N\theta$

For certain system parameters, the ray trajectory itself may be periodic. This occurs when $\theta/2\pi$ is a rational number

i.e., An integer N exists for which

$$N\theta = 2M\pi, M \in \mathbb{R} \quad (3.31)$$

From (3.11) & (3.31), we find $U_{N-1} = 0, U_{N-2} = -1$, whence from (3.10)

$$\begin{bmatrix} A & B \\ C & D \end{bmatrix}^N = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \quad (3.32)$$

→ the ray retraces its trajectory after traversing N unit cells.

***Application: Symmetric Confocal Resonator**

radii of curvature $R_1 = R_2 \equiv R = d$ — common focal point
 — see Fig. 3.3 a — mirror separation — $\therefore f_1 = f_2 = R/2$
 as in the identical lens waveguide

(3.7a) & (3.7d) yield $A = -1, D = -1$

Using the above, (3.12) gives the system parameter θ as

$$\theta = \cos^{-1} \left[\frac{1}{2}(A + D) \right] = \pi \quad (3.33)$$

Comparing (3.31) & (3.33), we see that, for $M = 1$ (a period in the lens waveguide being equivalent to a round trip in the resonator), we require $N = 2 \dots$

→ the rays retrace their paths after two round trips.

NB. This is independent of the initial launch condition (ρ_0, ρ'_0) .

3.3 Rays in Inhomogeneous Media

Trajectory governed by *Fermat's Principle of Least Time*

– expressed using the *calculus of variations* as...

$$\delta \int_{P_1}^{P_2} \underbrace{\left(\frac{n}{c} \right)}_{\text{Lagrangian}} ds = 0 \quad (3.34)$$

↑ functional ↓ minimize/maximize

i.e., Find the function that minimizes the functional

↑ a function of functions – a map from a vector space to its underlying scalar field

...where: P_1/P_2 – the start/end point

$n = n(\mathbf{r})$ – index of refraction, $\mathbf{r} = (x, y, z)$

ds – a differential element along the ray path

Here, since c (speed of light in vacuum) is constant, (3.34) is equivalent to requiring that the *optical path length* between the two points be *stationary* ...

a minimum $\longleftarrow \int_{P_1}^{P_2} n ds \longleftarrow$

Let us parametrize the ray trajectory as...

$$x = x(t), y = y(t), z = z(t) \quad (3.35)$$

where t is an arbitrary parameter. The differential element is then...

$$ds = \sqrt{(x')^2 + (y')^2 + (z')^2} dt \quad (3.36)$$

where the prime indicates differentiation with respect to t . The Lagrangian then becomes

$$L[t; x, y, z; x', y', z'] = n(x, y, z) \sqrt{(x')^2 + (y')^2 + (z')^2} dt \quad (3.37)$$

The functional is thus...

$$F[x, y, z] = \int_{t_1}^{t_2} L[t; x, y, z; x', y', z'] dt \quad (3.38)$$

The functional is stationary if the *Euler-Lagrange equations* are satisfied:

$$\frac{\partial L}{\partial x} = \frac{d}{dt} \left(\frac{\partial L}{\partial x'} \right), \quad \frac{\partial L}{\partial y} = \frac{d}{dt} \left(\frac{\partial L}{\partial y'} \right), \quad \frac{\partial L}{\partial z} = \frac{d}{dt} \left(\frac{\partial L}{\partial z'} \right) \quad (3.39)$$

Using (3.36), (3.39) may be rewritten as

$$\frac{\partial n}{\partial x} = \frac{d}{ds} \left(n \frac{dx}{ds} \right), \quad \frac{\partial n}{\partial y} = \frac{d}{ds} \left(n \frac{dy}{ds} \right), \quad \frac{\partial n}{\partial z} = \frac{d}{ds} \left(n \frac{dz}{ds} \right) \quad (3.40)$$

...or, in vector form, as:

$$\frac{d}{ds} \left(n \frac{d\mathbf{r}}{ds} \right) = \nabla n \quad (3.41)$$

The Ray Equation

*Ray Propagation in Lenslike Media

Lenslike – symmetry about the propagation direction z ...

$$n^2(\mathbf{r}) = n_0^2(1 - g^2\rho^2) \quad (3.42)$$

where: $\rho^2 = x^2 + y^2 \quad (3.43)$

n_0 – refractive index at $\rho = 0$

g – a constant characteristic of the medium

NB. $g^2\rho^2 \ll 1$

NB. Inspection of (3.42) reveals that a thin slab of such a medium must act as a thin convex lens, since the induced phase shift...

$$\frac{2\pi}{\lambda} n\Delta z \leftrightarrow \frac{2\pi}{\lambda} \frac{\rho^2}{2f}$$

...decreases as ρ^2 (e.g., expand n^2 to 1st order in ρ^2)

→ We have a...

quadratic index medium

≡ gradient index (GRIN) medium

Now, assume paraxial rays, $d/ds \rightarrow d/dz$, and $r = \rho$, ...then with (3.42) & (3.43), (3.41) approximates as...

$$\frac{d^2\mathbf{r}}{dz^2} + g^2\mathbf{r} = 0 \quad (3.44)$$

for $\mathbf{r} = (x, y, z) \rightarrow (x, y)$. Assuming meridional rays, (3.44) becomes

$$\frac{d^2\rho}{dz^2} + g^2\rho = 0 \quad (3.45)$$

Solving (3.45) for input position and slope ρ_0 & ρ'_0 , respectively, at $z = 0$ gives

$$\rho(z) = \rho_0 \cos(gz) + \left(\frac{\rho'_0}{g} \right) \sin(gz) \quad (3.46a)$$

$$\rho'(z) = -\rho_0 \sin(gz) + \rho'_0 \cos(gz) \quad (3.46b)$$

(3.46) can be written in matrix format as...

$$\begin{bmatrix} \rho \\ \rho' \end{bmatrix} = \begin{bmatrix} \cos(gz) & g^{-1} \sin(gz) \\ -g \sin(gz) & \cos(gz) \end{bmatrix} \begin{bmatrix} \rho_0 \\ \rho_0' \end{bmatrix} \quad (3.47)$$

(3.46a) implies sinusoidal oscillation of the ray through the optical axis, with a period given by

$$\text{(A is uppercase } \alpha \text{)} \quad \text{pitch} \xrightarrow{\uparrow} \text{A} = \frac{2\pi}{g} \quad (3.48)$$

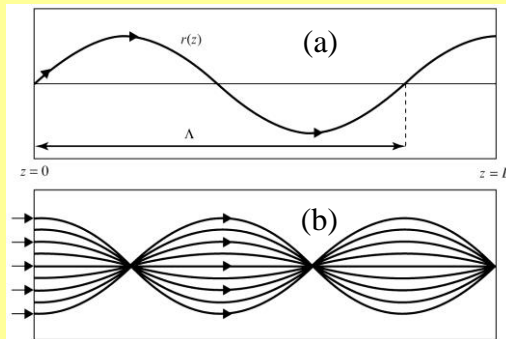


Fig 3.4: Path of rays in a medium with a quadratic index variation. From Yariv & Yeh, 6th ed., p. 76.

NB. Lenslike media can be used for imaging

“GRIN rod lenses”

Consider a bundle of parallel rays incident at $z = 0$ on a rod (as in Fig. 3.4a). From (3.46a)

$$\rho(z) = \rho_0 \cos(gz) \quad (3.49)$$

The rays will focus to a point of the axis after traversing a distance d such that

$$gd = \pi/2 \quad (3.50)$$

Since d is the focal length, (3.48) \rightarrow (3.50) yields

$$f = A/4 \quad (3.51)$$

- see Fig. 3.4b

Assuming such a rod in air, if its length L is not an integer multiple of f , there will be a distance h from the exit plane at which the rays will focus

$$h = \frac{1}{n_0 g \tan(gL)} \quad (3.52)$$

- see Problem 2.3, text

NB. Our derivation from (3.44) assumed meridional rays. For skew rays, a helical path about the z -axis will be taken – see Problem 2.19, text.

*Physical situation yielding quadratic index variation:

- (a) Propagation of laser beams with a Gaussian-like intensity profile in a lightly absorbing medium

$$\text{radial temperature variations } \frac{dn}{dT} > 0 \rightarrow \text{lenslike medium}$$

- (b) Propagation of laser beams with a Gaussian-like intensity profile in a *Kerr medium* \rightarrow *nonlinear response*

$$\begin{array}{c} \text{└─} n(\mathbf{r}) = n_0 + n_2 I(\mathbf{r}) \\ \text{Kerr coefficient ─┬─} \quad \text{┬─} \text{ beam intensity} \\ n_2 > 0 \rightarrow \text{lenslike medium} \end{array}$$