Media with Multiple Resonances

For a single resonance, we showed, in (2.15)

$$\alpha = \frac{e^2}{m\left(\omega_0^2 - \omega^2 + i\gamma\omega\right)}$$

This is the classical form of the corresponding quantum mechanical expression, determined from time-dependent perturbation theory as

$$\alpha = \frac{f_{21}e^2}{m\left(\omega_0^2 - \omega^2 + i\gamma\omega\right)} \qquad (2.53)$$

where

$$f_{21} = 2m\omega_0\hbar \,|< \mu_2 \,|\, x \,|\, \mu_1 >|^2 \qquad (2.54)$$

$$\int_v \mu_2^* x \mu_1 dV$$

oscillator strength (dimensionless)

Assuming $\mathbf{E} = E\hat{x}$

Recalling (1.84) & (2.9), for a dilute gas

$$\frac{\varepsilon}{\varepsilon_0} = 1 + \chi = 1 + \frac{N\alpha}{\varepsilon_0}$$

Thus, the more correct expression for ε, in terms of the relative strength of the transition, using (2.53) in the above, is

$$\frac{\varepsilon}{\varepsilon_0} = 1 + \frac{Ne^2}{m\varepsilon_0}\left\{\frac{f_{21}}{\omega_0^2 - \omega^2 + i\gamma\omega}\right\} \qquad (2.55)$$

This expression may be generalized to account for electron excitation to any level, where

$$\sum_{m=2}^{\infty} f_{m1} = 1 \qquad (2.56)$$

Sum Rule

energy level

Classically, assume the N atoms per unit volume have a fraction f_j of their electrons with resonance frequency ω_j and damping factor γ_j

The polarizability due to such an oscillation, following (2.53), is

$$\alpha_j = \frac{f_j e^2}{m\left(\omega_j^2 - \omega^2 + i\gamma_j\omega\right)} \qquad (2.57)$$

Assuming all oscillators are independent, the total polarizability is

$$\alpha = \sum_j \alpha_j \qquad (2.58)$$

(2.58) may be substituted into (2.9), for a dilute gas, or (2.48), for a dense medium, to find χ. The permittivity ε follows from (1.84).

There is also a sum rule for f_j:

$$\sum_j f_j = Z \qquad (2.59)$$

Number of electrons per atom

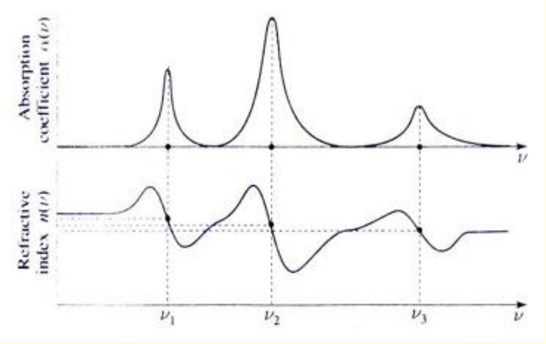

Fig. 2.7: Frequency dependence of the absorption coefficient and refractive index for a medium with three resonances. Saleh & Teich, 2nd ed., p. 179

Sellmeier Equation

Assuming multiple resonances that are well separated, and frequencies of interest that are far from such resonances such that (2.21)…

$$\chi(\omega) \approx \chi_0 \left\{ \frac{\omega_0^2}{\omega_0^2 - \omega^2} \right\}$$

…is valid for any individual resonance, the real refractive index may be approximated via (1.84), as

$$n^2 \approx 1+\sum_i \chi_{0i}\left\{\frac{\omega_i^2}{\omega_i^2-\omega^2}\right\}=1+\sum_i \chi_{0i}\left\{\frac{\lambda^2}{\lambda^2-\lambda_i^2}\right\} \qquad (2.60)$$

Sellmeier equation

→ χ_{01} & λ_i are fitting parameters

Table 2.1: Sellmeier equations for some materials at room temperature. Saleh & Teich, 2nd ed., p. 180

Material	Sellmeier Equation (Wavelength λ in μm)	Wavelength Range (μm)
Fused silica	$n^2 = 1 + \frac{0.6962\lambda^2}{\lambda^2-(0.06840)^2} + \frac{0.4079\lambda^2}{\lambda^2-(0.1162)^2} + \frac{0.8975\lambda^2}{\lambda^2-(9.8962)^2}$	0.21–3.71
Si	$n^2 = 1 + \frac{10.6684\lambda^2}{\lambda^2-(0.3015)^2} + \frac{0.0030\lambda^2}{\lambda^2-(1.1347)^2} + \frac{1.5413\lambda^2}{\lambda^2-(1104.0)^2}$	1.36–11
GaAs	$n^2 = 3.5 + \frac{7.4969\lambda^2}{\lambda^2-(0.4082)^2} + \frac{1.9347\lambda^2}{\lambda^2-(37.17)^2}$	1.4–11
BBO	$n_o^2 = 2.7359 + \frac{0.01878}{\lambda^2-0.01822} - 0.01354\lambda^2$	0.22–1.06
	$n_e^2 = 2.3753 + \frac{0.01224}{\lambda^2-0.01667} - 0.01516\lambda^2$	
KDP	$n_o^2 = 1 + \frac{1.2566\lambda^2}{\lambda^2-(0.09191)^2} + \frac{33.8991\lambda^2}{\lambda^2-(33.3752)^2}$	0.4–1.06
	$n_e^2 = 1 + \frac{1.1311\lambda^2}{\lambda^2-(0.09026)^2} + \frac{5.7568\lambda^2}{\lambda^2-(28.4913)^2}$	
$LiNbO_3$	$n_o^2 = 2.3920 + \frac{2.5112\lambda^2}{\lambda^2-(0.217)^2} + \frac{7.1333\lambda^2}{\lambda^2-(16.502)^2}$	0.4–3.1
	$n_e^2 = 2.3247 + \frac{2.2565\lambda^2}{\lambda^2-(0.210)^2} + \frac{14.503\lambda^2}{\lambda^2-(25.915)^2}$	

Conductive Media – *the electron gas*

metals, semiconductors, ionized gases…

free electric charges

→ electric current density: $\mathcal{J}(\mathbf{r}, t) \rightarrow \mathbf{J}(\mathbf{r})$

Recall Maxwell's equation (1.80):

$$\nabla\times\mathbf{H} = j\omega\mathbf{D}+\mathbf{J}$$

Now, assume the medium has linear conductive properties:

$$\mathbf{J} = \sigma\mathbf{E} \qquad (2.61)$$

conductivity

(2.61) → (1.80), using (1.75) yields

$$\nabla\times\mathbf{H} = j\omega\varepsilon_e\mathbf{E} \qquad (2.62)$$

where…

$$\varepsilon_e = \varepsilon + \frac{\sigma}{j\omega} \qquad (2.63)$$

effective permittivity

NB. Contribution of σ to ε_e decreases as ω increases.

The only change in our earlier analysis of wave transport that now needs to be made to accommodate free charge is to let $\varepsilon \longrightarrow \varepsilon_e$, everywhere.

Hence, (1.99) & (1.116) become:

$$k = \beta - j\,{}^{\alpha}\!/_{2} = \omega\sqrt{\varepsilon_e \mu_0} \qquad (2.64)$$

…and (1.101) becomes:

$$n - \frac{j\alpha}{2k_0} = \sqrt{\varepsilon_e / \varepsilon_0} \qquad (2.65)$$

… and (1.102) becomes:

$$\eta = \sqrt{\mu_0 / \varepsilon_e} \qquad (2.66)$$

When $\sigma/\omega >> \varepsilon$, conductive effects dominate

$$\varepsilon_e \approx \sigma / i\omega \qquad (2.67)$$

Using (2.67), (2.65) then approximate as

$$n - \frac{i\alpha}{2k_0} \approx \sqrt{\frac{\sigma}{j\omega\varepsilon_0}} = (1-i)\sqrt{\frac{\sigma}{2\omega\varepsilon_0}} \qquad (2.68)$$

Hence,

$$n \approx \sqrt{\frac{\sigma}{2\omega\varepsilon_0}} \qquad (2.69)$$

and, using (84),

$$\alpha \approx 2k_0\sqrt{\frac{\sigma}{2\omega\varepsilon_0}} = \sqrt{2\omega\sigma\mu_0} \qquad (2.70)$$

Likewise, (2.66) approximates as

$$\eta \approx \sqrt{\frac{j\omega\mu_0}{\sigma}} = (1+j)\sqrt{\frac{\omega\mu_0}{2\sigma}} \qquad (2.71)$$

From (2.70), we can define

$$\delta = \frac{1}{\alpha} \approx \frac{1}{\sqrt{2\omega\sigma\mu_0}} \qquad (2.72)$$

skin depth – distance into a conductor that a field transverses before its intensity is reduced by e^{-1}.

The Drude Model

J – arises from some average motion of charges (of charge q)…

$$\mathbf{J} = N q \mathbf{v} \qquad (2.73)$$

Number density (N) *average charge velocity* ($\mathbf{v}$)

In the presence of a field: $\mathbf{v} \neq 0$ (no field: $\mathbf{v} = 0$)

…determined by the mean time τ between "collisions"

$\tau \leftarrow$ *relaxation time*

Assume a static ($\omega \to 0$) field → constant acceleration of q

$$\mathbf{v} = \mathbf{a}\tau = \mathbf{F}\tau / m = q\mathbf{E}\tau / m \qquad (2.74)$$

charge mass (m)

(2.74) → (2.73), compared to (2.61), yields

$$\sigma_0 = \frac{Nq^2\tau}{m} \qquad (2.75)$$

DC conductivity (σ_0)

Let us consider time dependent fields. We can express (2.73) as…

$$\mathbf{J}(t) = Nq\mathbf{P}(t) / m \qquad (2.76)$$

…where **P**(t) is the *average charge momentum* (not the polarization)

A random charge, at time t, will experience a collision before time $t + \Delta t$, with a probability of $\Delta t/\tau$

→ probability of avoiding a collision:

$1 - \Delta t/\tau$

During Δt, the charge momentum evolves under the force **F** of the applied field:

$$\mathbf{P}(t+\Delta t) \approx \left(1-\frac{\Delta t}{\tau}\right)\left\{\mathbf{P}(t)+\mathbf{F}(t)\Delta t\right\}$$

$$= \mathbf{P}(t) - \frac{\Delta t}{\tau}\mathbf{P}(t) + \mathbf{F}(t)\Delta t \qquad (2.77)$$

to 1st order in Δt

$$\Rightarrow \mathbf{P}(t+\Delta t) - \mathbf{P}(t) = \frac{-\Delta t}{\tau}\mathbf{P}(t) + \Delta t\mathbf{F}(t) \qquad (2.78)$$

In limit $\Delta t \rightarrow 0$, (2.78) yields

$$\frac{d\mathbf{P}}{dt}(t) = \frac{-\mathbf{P}(t)}{\tau} + \mathbf{F}(t) \qquad (2.79)$$

Equation of motion for charge momentum

Assuming a time-harmonic field, see (1.99), we will then have

$$\mathbf{P}(t) = \mathrm{Re}\left\{\mathbf{P}(\omega)e^{i\omega t}\right\} \qquad (2.80)$$

Thus, (2.79) becomes

$$i\omega\mathbf{P}(\omega) = -\frac{\mathbf{P}(\omega)}{\tau} + q\mathbf{E}(\omega)$$

$$\Rightarrow \mathbf{P}(\omega) = \frac{\tau q\mathbf{E}(\omega)}{1+i\omega\tau} \qquad (2.81)$$

(2.81) → (2.76), in the frequency domain, yields…

$$\mathbf{J}(\omega) = \frac{Nq\tau\mathbf{E}(\omega)}{m(1+i\omega\tau)} = \left(\frac{\sigma_0}{1+i\omega\tau}\right)\mathbf{E}(\omega)$$

$$\Rightarrow \mathbf{J}(\omega) = \sigma(\omega)\mathbf{E}(\omega) \tag{2.82}$$

where

$$\sigma(\omega) = \frac{\sigma_0}{1+i\tau\omega} \tag{2.83}$$

(2.83) ⟶ (2.63) gives the effective permittivity as

$$\varepsilon_e = \varepsilon + \frac{\sigma_0}{i\omega(1+i\omega\tau)} \tag{2.84}$$

For $\omega >> 1/\tau$, (2.84) becomes

$$\varepsilon_e \approx \varepsilon - \frac{\sigma_0}{\tau\omega^2} \tag{2.85}$$

NB. Conductivity acts to depress Re$\{\varepsilon_e\}$ at high frequencies

If $\varepsilon = \varepsilon_0$ (dilute gas), then (2.85) may be written as

$$\varepsilon_e = \varepsilon_0\left(1 - \omega_p^2 \big/ \omega^2\right) \tag{2.86}$$

where

$$\omega_p = \sqrt{\frac{\sigma_0}{\varepsilon_0\tau}} \tag{2.87}$$

plasma frequency

(2.75) ⟶ (2.87) yields (2.51), for $q = -e$

$$\omega_p = \sqrt{\frac{Nq^2}{\varepsilon_0 m}} \tag{2.88}$$

NB. (2.86) may also be found from (2.11) for $\omega_0 = 0$, $\gamma = 0$

Rewriting (2.86) as

$$\omega = \frac{\omega_p}{\sqrt{1 - \varepsilon_e / \varepsilon_0}} \qquad (2.89)$$

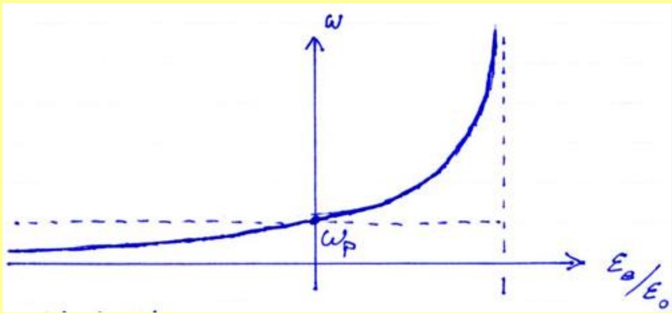

Fig. 2.8: Frequency dependence of the relative permittivity in the Drude model.

$\omega < \omega_p$ $(\varepsilon_e < 0)$

$\omega > \omega_p$ $(\varepsilon_e > 0)$

- For $\omega < \omega_p$, $\varepsilon_e < 0$ and k is then pure imaginary; (2.86) → (2.64) yields

$$\frac{-i\alpha}{2} = \omega\sqrt{\mu_0\varepsilon_0\left(1 - \frac{\omega_p^2}{\omega^2}\right)} = k_0\sqrt{1 - \frac{\omega_p^2}{\omega^2}}$$

$$\Rightarrow \alpha = 2k_0\sqrt{\frac{\omega_p^{\;2}}{\omega^2} - 1} \qquad (2.90)$$

NB. α decreases as ω increases, vanishing at $\omega = \omega_p$

Rewriting (2.90) as

$$\omega = \frac{2k_0\omega_p}{\sqrt{\alpha^2 + 4k_0^2}} \qquad (2.91)$$

In this frequency region…

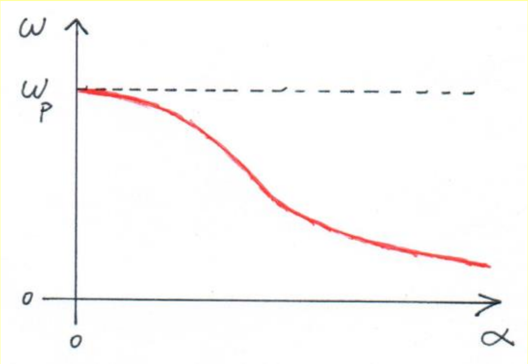

Fig. 2.9: Frequency dependence of the attenuation in the Drude model for $\omega < \omega_p$.

- For $\omega > \omega_p$, $\varepsilon_e > 0$ and k is then purely real; (2.86) → (2.64) then yields

$$\beta = \omega\sqrt{\mu_0\varepsilon_0\left(1-\frac{\omega_p^2}{\omega^2}\right)} = k_0\sqrt{1-\frac{\omega_p^2}{\omega^2}} \qquad (2.92)$$

Now the conductive medium behaves like a lossless dielectric… but with different dispersion characteristics.

Rewriting (2.92) as

$$\omega = \sqrt{\omega_p^2 + c^2\beta^2} \qquad (2.93)$$

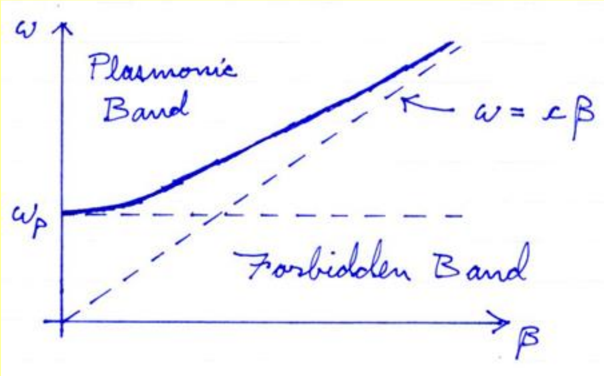

Fig. 2.10: Frequency dependence of the propagation constant in the Drude model for $\omega > \omega_p$.

At $\omega = \omega_p$, $\beta = 0$ → wave does not travel

↳ Free charge undergoes longitudinal oscillations…

>> *Plasmon* <<

Comparing (1.86) & (2.92) reveals

$$n = \sqrt{1 - \omega_p^2 / \omega^2} \qquad (2.94)$$

Rewriting (2.94):

$$\omega = \frac{\omega_p}{\sqrt{1 - n^2}} \qquad (2.95)$$

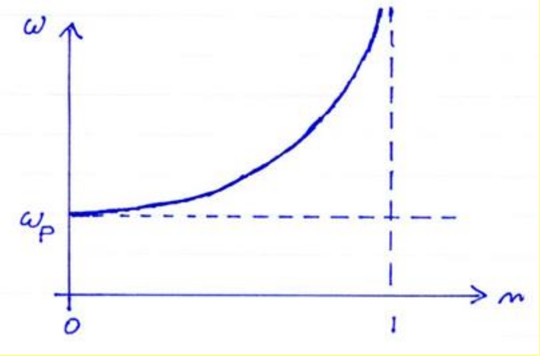

Fig. 2.11: Frequency dependence of the index of refraction in the Drude model for $\omega > \omega_p$.