1.7 Plane Waves in Uniaxially Anisotropic Media

- uniaxial media are worth special attention since they are common in photonics

(1.229) can be rewritten – or derived from (1.221) – as…

$$\left(\frac{k^2}{n_o^2}-\frac{\omega^2}{c^2}\right)\left(\frac{k_x^2+k_y^2}{n_e{}^2}+\frac{k_z{}^2}{n_o^2}-\frac{\omega^2}{c^2}\right)=0 \qquad (1.233)$$

relation between ω and **k** for the ordinary wave

relation for the extraordinary wave

*Let θ be the angle between the direction of propagation and the crystal's c-axis. (1.233) gives the eigen-indices as…

O wave: $n = n_o$ (1.234)

E wave: $\dfrac{1}{n^2}=\dfrac{\sin^2\theta}{n_e^2}+\dfrac{\cos^2\theta}{n_o^2}$ (1.235)

NB. If $\theta = 0$ (propagation along c-axis), the eigen-indices of both modes become degenerate.

How to determine eigenvectors?

→ Can't use (1.225) for the O wave because denominators vanish.

→ Must return to matrix equation (1.220).

Rewriting it using $\varepsilon_x = \varepsilon_y = \varepsilon_0 n_o^2$, $\varepsilon_z = \varepsilon_0 n_e^2$ and $\mathbf{k}_0 = \omega n_o \mathbf{s}/c$ (or using (1.219) to re-derive the matrix equation) yields…

$$\begin{bmatrix} s_x^2 & s_x s_y & s_x s_z \\ s_x s_y & s_y^2 & s_y s_z \\ s_x s_z & s_y s_z & (n_e/n_o)^2-\left(s_x^2+s_y^2\right) \end{bmatrix}\begin{bmatrix} E_x \\ E_y \\ E_z \end{bmatrix}=0 \qquad (1.236)$$

Inspection reveals…

$$\mathbf{E}_o = (s_y, -s_x, 0) \qquad (1.237)$$

We CAN use (1.225) for the E wave…

$$\mathbf{E}_e = \left(\frac{s_x}{n^2 - n_o^2}, \frac{s_y}{n^2 - n_o^2}, \frac{s_z}{n^2 - n_e^2} \right) \qquad (1.238)$$

… for n given by (1.235), $\mathbf{k}_e = \omega n\, \mathbf{s}/c$

NB. s_x, s_y, s_z are direction cosines

NB. For the O wave, $\mathbf{E}\cdot\mathbf{k}_o = 0$
For the E wave, $\mathbf{E}\cdot\mathbf{k}_e \neq 0$ – see (1.226d) – because there is a small angle between **E** & **D** $\rightarrow$ *nearly* transverse

NB. Since $\mathbf{D}\cdot\mathbf{k} = 0$, unit vector parallel to the *c*-axis

O wave: $$\mathbf{D}_0 = \frac{\mathbf{k} \times \hat{\mathbf{c}}}{|\mathbf{k} \times \hat{\mathbf{c}}|} \qquad (1.239)$$

E wave: $$\mathbf{D}_e = \frac{\mathbf{D} \times \mathbf{k}_e}{|\mathbf{D} \times \mathbf{k}_e|} \qquad (1.240)$$

*Let us rewrite the propagation direction **s** in spherical coordinates

azimuth angle

$$\hat{\mathbf{s}} = (\sin\theta\cos\phi, \sin\theta\sin\phi, \cos\theta) \qquad (1.241)$$

polar angle

…and then rewrite the normal modes for **E**:

O wave: $$\mathbf{E}_o = (\sin\phi, -\cos\phi, 0) \qquad (1.242)$$

For the E wave, (1.235) $\rightarrow$ (1.238) and partial normalization yields…

$$\mathbf{E}_e = \left(n_e^2\cos\theta\cos\phi, n_e^2\cos\theta\sin\phi, -n_o^2\sin\theta\right) \qquad (1.243)$$

NB. $\mathbf{E}_o \cdot \mathbf{E}_e = 0$ $\leftarrow$ this is true for isotropic & uniaxial material only – see the general orthogonality relations

*The normal modes for the electric displacement be found, using (1.210), as…

O wave: $\mathbf{D}_o = (\sin\phi, -\cos\phi, 0)$ (1.244)

E wave: $\mathbf{D}_e = (\cos\theta\cos\phi, \cos\theta\sin\phi, -\sin\theta)$ (1.245)

*Polarized light propagating with direction **s** in a uniaxial medium can always be written as a linear combination of normal modes…

$$\mathbf{D} = C_o\mathbf{D}_o e^{-i\mathbf{k}_o\cdot\mathbf{r}} + C_e\mathbf{D}_e e^{-i\mathbf{k}_e\cdot\mathbf{r}} \quad (1.246)$$

phase retardation between these two components due to their difference in phase velocity

New polarization state

C_o, C_e – constants

NB. For a parallel plate of thickness d, with the z-axis perpendicular to the plate surfaces…

$$\Gamma = (k_{ez} - k_{oz})d \quad (1.247)$$

phase retardation

z-components

Optical Rotary Power & Faraday Rotation

Media exist for which normal modes are "circularly polarized"

→ Circularly birefringent ⇔ Optically active

For linearly polarized light, its plane of polarization rotates as it traverses the medium.

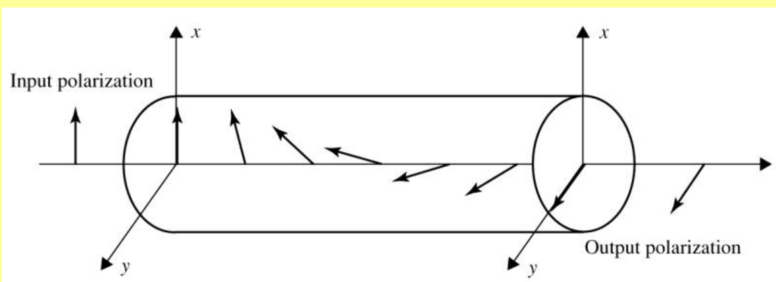


Fig 1.17: Rotation of the plane of polarization by an optically active medium. From Yariv & Yeh, 6th ed., p.39.

*Rotary power (degrees/mm@λ) due to…

- in solids: crystalline structure (space group)
- in liquids: asymmetric molecular structure

└→ *i.e.* helical structure

└→ almost all have mirror-image isomers with opposite rotary power.

→ chiral form: LH = levo (L)
RH = dextro (D)

*Sense of rotation…

- Fixed in relation to direction of propagation, **k**

└→ a reflection back through the medium gives zero net rotation

i.e. the medium is *reciprocal*

- Viewed by an observer facing into the incident beam…

Dextrorotary (RH) media

└→ CCW rotation of the plane of polarization

Levorotary (LH) media

└→ CW rotation…

*Let n_r & n_l be the refractive indices associated with the right & left circularly polarized eigenvalues, respectively.

The rotary power has been found to be…

$$\rho = \frac{\pi}{\lambda}\left(n_l - n_r\right) \qquad (1.248)$$

Optical rotation is CCW (RH) if $\rho > 0$, CW (LH) if $\rho < 0$

└→ Plane of polarization turns in the same sense as the circulatory polarized wave with the greater phase velocity.

	λ (Å)	ρ (degree/mm)
Quartz	4000	49
	4550	37
	5000	31
	5500	26
	6000	22
	6500	17
$AgGaS_2$	4850	950
	4900	700
	4950	600
	5000	500
	5050	430
Se	7500	180
	10,000	30
Te	(6 μm)	40
	(10 μm)	15
TeO_2	3698	587
	4382	271
	5300	143
	6328	87
	10,000	30

Table 1.3: Optical Rotary Power of Some Solids. From Yariv & Yeh, 6^{th} ed., p.39.

Faraday Rotation

→ The rotation of the plane of polarization when linearly polarized light traverses a medium (usually isotropic) in a strong magnetic field

Faraday effect

originates from the effect of the static magnetic field on the motions of electrons via the Lorentz force

- Atomic view similar to Zeeman effect…

→ magnetic field causes energy level to be split into sublevels

→ conservation of angular momentum requires RHC & LHC light to interact with different sublevels.

*For a Faraday cell, the rotation per unit length is…

$$\rho = V\,B \qquad (1.249)$$

specific rotation — Verdet constant — magnetic field component in the direction of propagation

Nonreciprocal optical devices…

→ isolators & circulators …use the Faraday effect

1.8 The Jones Matrix Method

→ a systematic method to treat propagation through a series of birefringent elements

*In birefringent media…

→ either biaxial or uniaxial, most commonly the latter

…light propagation consists of a linear superposition of *normal modes*

- well defined phase velocities & polarization direction
  - fixed by the medium
  - denoted by slow (s) & fast (f) axes (for that propagation direction)

NB. Retardation plates (aka: wave plates)…

– cut so that the c-axis lies in the plane of the plate surfaces.

– normally incident light propagates perpendicular to c-axis

→ These are *polarization state converters*

i.e., any incident polarization state can be converted to any other polarization state using a suitable wave plate

Jones Matrix Formalism

- assume total transmission

*The polarization state of the incident beam is described by

$$\mathbf{V} = \begin{bmatrix} V_x \\ V_y \end{bmatrix} \qquad (1.250)$$

complex numbers

Let the lab system be denoted by the x & y axes. We need to decompose the beam into a linear combination of fast & slow axes to show how light propagates in the wave plate.

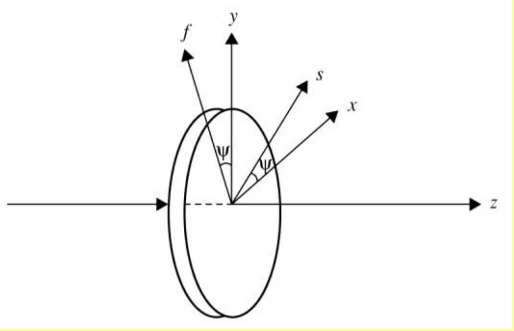


Fig. 1.18: A wave plate with azimuth angle ψ, defined as the angle between the s axis (plate, or s-f coordinate system) and the x axis (lab, or x-y coordinate system). From Yariv & Yeh, 6th ed., p.44.

$$\begin{bmatrix} V_s \\ V_f \end{bmatrix} = \begin{bmatrix} \cos\psi & \sin\psi \\ -\sin\psi & \cos\psi \end{bmatrix} \begin{bmatrix} V_x \\ V_y \end{bmatrix} = R(\psi) \begin{bmatrix} V_x \\ V_y \end{bmatrix} \qquad (1.251)$$

Slow & fast components of **V**

coordinate rotation matrix

Due to the difference in phase velocities, one component is retarded with respect to the other

changes the state of the emerging beam

*Let n_s & n_f be the indices associated with the slow & fast components, respectively. In the s-f coordinate system, the polarization state of the emerging beam is…

$$\begin{bmatrix} V_s' \\ V_f' \end{bmatrix} = \begin{bmatrix} e^{-in_s 2\pi d/\lambda} & 0 \\ 0 & e^{-in_f 2\pi d/\lambda} \end{bmatrix} \begin{bmatrix} V_s \\ V_f \end{bmatrix} \qquad (1.252)$$

…where d is the plate thickness & λ is the wavelength.

The relative change in phase is the difference in the diagonal exponents…

$$\underbrace{\Gamma}_{retardation} = \frac{2\pi}{\lambda}(n_s - n_f)d \qquad (1.253)$$

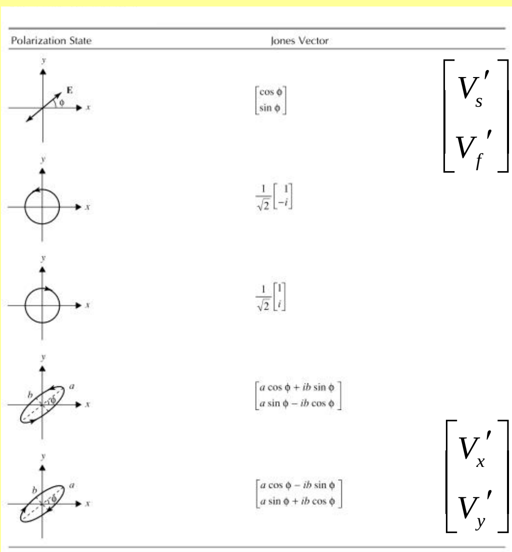
NB. Typically, the birefringence is small…

i.e., For $|n_s - n_f| << n_s, n_f$

$$\underbrace{\phi}_{mean\ absolute\ phase\ change} = \frac{1}{2}\left(n_s + n_f\right)2\pi d / \lambda \qquad (1.254)$$

$\rightarrow \phi >> \Gamma$

(1.252) may be rewritten as…



Polarization State	Jones Vector
	$\begin{bmatrix} \cos\phi \\ \sin\phi \end{bmatrix}$
	$\frac{1}{\sqrt{2}}\begin{bmatrix} 1 \\ -i \end{bmatrix}$
	$\frac{1}{\sqrt{2}}\begin{bmatrix} 1 \\ i \end{bmatrix}$
	$\begin{bmatrix} a\cos\phi + ib\sin\phi \\ a\sin\phi - ib\cos\phi \end{bmatrix}$
	$\begin{bmatrix} a\cos\phi - ib\sin\phi \\ a\sin\phi + ib\cos\phi \end{bmatrix}$

Table 1.4: Jones Vectors. From Yariv & Yeh, 6th ed., p.44.

$$\begin{bmatrix} V_s' \\ V_f' \end{bmatrix} = e^{-i\phi}\begin{bmatrix} e^{-i\Gamma/2} & 0 \\ 0 & e^{+i\Gamma/2} \end{bmatrix}\begin{bmatrix} V_s \\ V_f \end{bmatrix} \qquad (1.255)$$

In the x-y coordinate system, the Jones vector of the polarization state of the emerging beam is found by transforming from the s-f to the x-y coordinate system…

$$\begin{bmatrix} V_x' \\ V_y' \end{bmatrix} = \begin{bmatrix} \cos\psi & -\sin\psi \\ \sin\psi & \cos \end{bmatrix}\begin{bmatrix} V_s' \\ V_f' \end{bmatrix} \qquad (1.256)$$

Thus, the transformation by the retardation plate is found by (1.251) → (1.255) → (1.256) as…

$$\begin{bmatrix} V_x' \\ V_y' \end{bmatrix} \equiv R(-\psi)W_0R(\psi)\begin{bmatrix} V_x \\ V_y \end{bmatrix} \qquad (1.257)$$

where: $$R(\psi) = \begin{bmatrix} \cos\psi & \sin\psi \\ -\sin\psi & \cos\psi \end{bmatrix} \qquad (1.258)$$

coordinate rotation matrix

and: $$W_0 = e^{-i\phi}\begin{bmatrix} e^{-i\Gamma/2} & 0 \\ 0 & e^{+i\Gamma/2} \end{bmatrix} \qquad (1.259)$$

Jones matrix for the retardation plate

Hence, from (1.257), the retardation plate is represented by the product of three matrices…

$$W = R(-\psi)W_0R(\psi) \qquad (1.260)$$

NB. $W^{\dagger}W = 1$ →Hermetian conjugate

Unitary transformation…

physical properties often invariant under such transformation

e.*g*., mutually orthogonal beams entering a waveplate will still be orthogonal leaving the waveplate

*For an ideal homogeneous linear sheet polarizer, transmission axis oriented parallel to the lab *x*-axis, the Jones matrix is

$$P_0 = e^{-i\phi}\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} \qquad (1.261)$$

where ϕ is the absolute phase accumulated.

└──→ this is an anisotropic medium…

→ one of the normal modes is attenuated by absorption

*For a polarizer rotated by an angle ψ about the z-axis, the Jones matrix is

$$P = R(-\psi)P_0R(\psi) \qquad (1.262)$$

*Neglecting the absolute phase, the Jones matrix forms for polarizers transmitting light with electric field vectors along the x & y directions are…

$$P_x = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} \quad \& \quad P_y = \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} \qquad (1.263)$$

*The Jones vector for a beam of light emerging from a train of retardation plates & polarizers is found by matrix multiplying the sequence of Jones matrices.

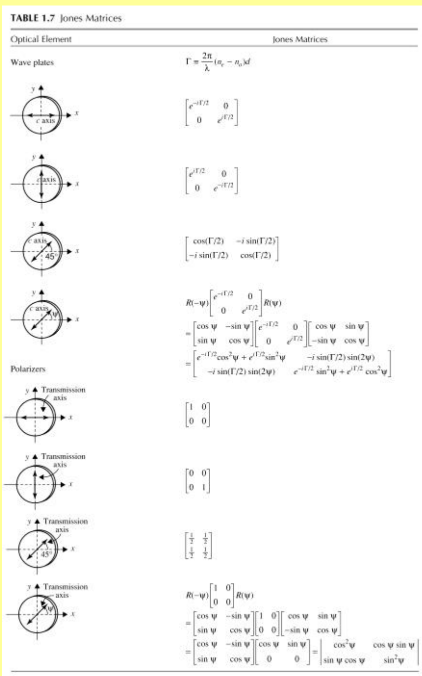


TABLE 1.7 Jones Matrices

Optical Element	Jones Matrices
Wave plates	$\Gamma = \frac{2\pi}{\lambda}(n_e - n_o)d$
c axis	$\begin{bmatrix} e^{-i\Gamma/2} & 0 \\ 0 & e^{i\Gamma/2} \end{bmatrix}$
c axis	$\begin{bmatrix} e^{i\Gamma/2} & 0 \\ 0 & e^{-i\Gamma/2} \end{bmatrix}$
c axis, 45°	$\begin{bmatrix} \cos(\Gamma/2) & -i\sin(\Gamma/2) \\ -i\sin(\Gamma/2) & \cos(\Gamma/2) \end{bmatrix}$
c axis, ψ	$R(-\psi)\begin{bmatrix} e^{-i\Gamma/2} & 0 \\ 0 & e^{i\Gamma/2} \end{bmatrix}R(\psi) = \begin{bmatrix} \cos\psi & -\sin\psi \\ \sin\psi & \cos\psi \end{bmatrix}\begin{bmatrix} e^{-i\Gamma/2} & 0 \\ 0 & e^{i\Gamma/2} \end{bmatrix}\begin{bmatrix} \cos\psi & \sin\psi \\ -\sin\psi & \cos\psi \end{bmatrix} = \begin{bmatrix} e^{-i\Gamma/2}\cos^2\psi + e^{i\Gamma/2}\sin^2\psi & -i\sin(\Gamma/2)\sin(2\psi) \\ -i\sin(\Gamma/2)\sin(2\psi) & e^{-i\Gamma/2}\sin^2\psi + e^{i\Gamma/2}\cos^2\psi \end{bmatrix}$
Polarizers	
Transmission axis	$\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$
Transmission axis	$\begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}$
Transmission axis, 45°	$\begin{bmatrix} \frac{1}{2} & \frac{1}{2} \\ \frac{1}{2} & \frac{1}{2} \end{bmatrix}$
Transmission axis, ψ	$R(-\psi)\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}R(\psi) = \begin{bmatrix} \cos\psi & -\sin\psi \\ \sin\psi & \cos\psi \end{bmatrix}\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}\begin{bmatrix} \cos\psi & \sin\psi \\ -\sin\psi & \cos\psi \end{bmatrix} = \begin{bmatrix} \cos\psi & -\sin\psi \\ \sin\psi & \cos\psi \end{bmatrix}\begin{bmatrix} \cos\psi & \sin\psi \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} \cos^2\psi & \cos\psi\sin\psi \\ \sin\psi\cos\psi & \sin^2\psi \end{bmatrix}$

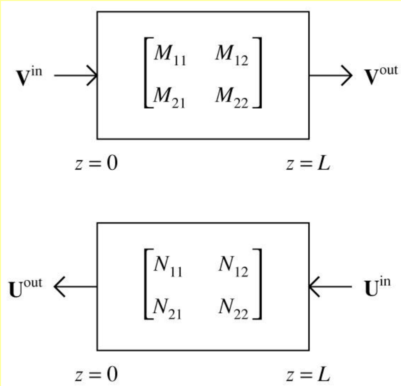
Table 1.5: Jones Matrices. From Yariv & Yeh, 6th ed., p.46.

General Properties of the Jones Matrix

- useful for understanding the transmission properties of birefringent networks

→ Consider a general birefringent system…

…a series of wave plates whose optical axes are arbitrarily oriented



We define the respective Jones matrices and consider their fundamental properties:

Fig. 1.9: Schematic drawing of input–output relationships and definition of Jones matrices M and N. Yariv & Ye, 6th ed., p. X

(1) Incidence from the left

$\mathbf{V}^{\text{in}}$ – input Jones vector at $z = 0$

$\mathbf{V}^{\text{out}}$ – output Jones vector at $z = L$

$$\begin{bmatrix} V_x^{out} \\ V_y^{out} \end{bmatrix} = \begin{bmatrix} M_{11} & M_{12} \\ M_{21} & M_{22} \end{bmatrix} \begin{bmatrix} V_x^{in} \\ V_y^{in} \end{bmatrix} \equiv M \begin{bmatrix} V_x^{in} \\ V_y^{in} \end{bmatrix} \qquad (1.264)$$

(2) Incidence from the right

$\mathbf{U}^{\text{in}}$ – input Jones vector at $z = L$

$\mathbf{U}^{\text{out}}$ – output Jones vector at $z = 0$

$$\begin{bmatrix} U_x^{out} \\ U_y^{out} \end{bmatrix} = \begin{bmatrix} N_{11} & N_{12} \\ N_{21} & N_{22} \end{bmatrix} \begin{bmatrix} U_x^{in} \\ U_y^{in} \end{bmatrix} \equiv N \begin{bmatrix} U_x^{in} \\ U_y^{in} \end{bmatrix} \qquad (1.265)$$

NB. We assume here that the light path retraces that of case (1).

Time Reversal Symmetry

- let time be reversed

output beam will retrace the beam path

$\rightarrow$ beam becomes the phase conjugate of the input

Therefore: $\mathbf{U}^{in} = (\mathbf{V}^{out})^*$ Likewise: $\mathbf{U}^{out} = (\mathbf{V}^{in})^*$

Thus: $NM^* = 1$ (1.266)

Principle of Reciprocity

Via reciprocity, (1.264) & (1.265), we have

$$N = M \tag{1.267}$$

transpose of M

i.e., $N_{11} = M_{11}$ $N_{21} = M_{12}$

$N_{22} = M_{22}$ $N_{12} = M_{21}$

Using (1.266) & (1.267), it can be shown that.

$$N^\dagger N = 1 \quad \& \quad M^\dagger M = 1 \tag{1.268}$$

Hermetian conjugate of N

i.e., N & M are unitary matrices

*If M is written as…

$$M = \begin{bmatrix} A & B \\ C & D \end{bmatrix} \tag{1.269}$$

…then the inverse of M is

$$M^{-1} = M^\dagger = \begin{bmatrix} A^* & C^* \\ B^* & D^* \end{bmatrix} \tag{1.270}$$

Furthermore, since (1.260) demonstrates that the Jones matrix is unimodular…

$$\det(M) = AD - BC = 1 \qquad (1.271)$$

…the inverse of the Jones matrix can also be written as

$$M^{-1} = \begin{bmatrix} D & -B \\ -C & A \end{bmatrix} \qquad (1.272)$$

Using (1.270) & (1.272), we find…

$$C = -B^* \quad \& \quad D = A^* \qquad (1.273)$$

…allowing the Jones matrix to be written as

$$M = \begin{bmatrix} A & B \\ -B^* & A^* \end{bmatrix} \qquad (1.274)$$