Poincaré Sphere

Stokes parameters – introduced to describe partially polarized light

→ useful also to describe polarization states of polarized light

e.g. in optical birefringent networks, such as those containing waveplates; also optical fibres where PMD is present.

*Setting $S_0 = 1$, from (1.199) we note that for polarized light all (S_1, S_2, S_3) sit on the "surface of a unit sphere"

Poincaré sphere

Each point on the sphere represents a unique polarization state

e.g. (0, 0, 1) → LHCP   (0, 0, –1) → RHCP

"North Pole"   "South Pole"

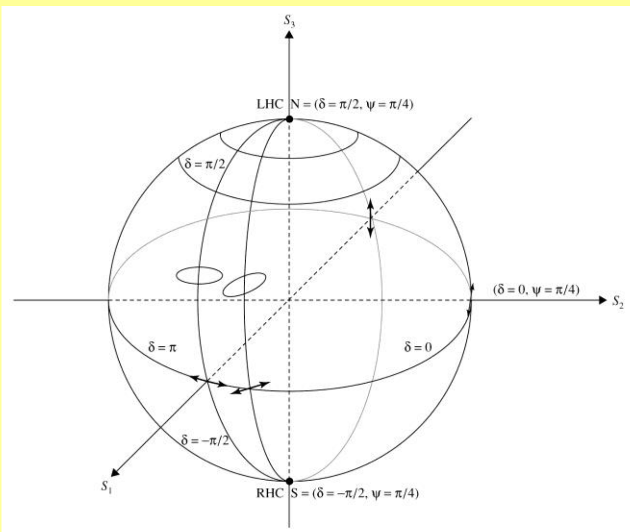


Fig. 1.11: The Poincaré Sphere (showing polarization states at various points).

- Equator: each point is a unique linear polarization *e.g.* (1, 0, 0) is horizontal; (– 1, 0, 0) is vertical
- All the rest are elliptical PS

NB. Any pair of antipodal points describe states with orthogonal polarization.

*Comparing (1.181) & the ratio of (1.202c) and (1.202b), we find

$$\tan(2\phi) = S_2/S_1 \qquad (1.203)$$

ϕ – *inclination angle*

Similarly, comparing (1.182) & (1.202d) shows

$$\sin(2\theta) = -S_3 \qquad (1.204)$$

θ – *ellipticity angle* – see (1.179)

NB. S_2/S_1 = *constant* is a vertical plane containing the poles.

BUT: As S_1 and S_2 are confined to the sphere surface, S_2/S_1 = constant represents a meridian (longitude)

$\therefore$ (1.203) $\Rightarrow \phi$ is constant on a meridian

$\rightarrow$ a class of elliptical polarization states with the same inclination angle, but with different ellipticities.

NB. S_3 = *constant* is a horizontal plane

a circle on the sphere parallel to the equatorial plane (a latitude)

$\therefore$ (1.204) $\Rightarrow \theta$ is constant on a latitude

$\rightarrow$ a class of elliptical polarization states with the same ellipticity, but with different inclination angles.

*Now, consider two points on the Poincaré sphere…

$$\mathbf{S}_a = (1, S_{a1}, S_{a2}, S_{a3}) \qquad (1.205a)$$

$$\mathbf{S}_b = (1, S_{b1}, S_{b2}, S_{b3}) \qquad (1.205b)$$

Using (1.195) & (1.202), it may be seen that

$$\mathbf{S}_a \cdot \mathbf{S}_b = 2\,|\,\mathbf{J}_a{}^* \cdot \mathbf{J}_b\,| \qquad (1.206)$$

corresponding Jones vectors

*Since S_0 = 1 for polarized light, it is sufficient to define only 3-component unit vectors. These Stokes vectors, such as

$$\mathbf{s}_a = \left(S_{a1}, S_{a2}, S_{a3}\right) \tag{1.207a}$$

$$\mathbf{s}_b = \left(S_{b1}, S_{b2}, S_{b3}\right) \tag{1.207b}$$

locate points on the Poincaré sphere. (1.206) may thus be rewritten as

$$|\,\mathbf{J}_a \cdot \mathbf{J}_b\,|^2 = \frac{1}{2}\mathbf{S}_a \cdot \mathbf{S}_b = \frac{1}{2}\left(1 + \mathbf{s}_a \cdot \mathbf{s}_b\right) \tag{1.208}$$

NB. For antipodal points

$$\mathbf{s}_a \cdot \mathbf{s}_b = -1 \tag{1.209}$$

1.6 Electromagnetic Propagation in Crystals

Anisotropic media

└→ propagation determined by *dielectric tensor* **ε**

(1.1) is represented as

$$D_i = \varepsilon_{ij} E_j \tag{1.210}$$

└ summation convention (repeated indices)

For non-absorbing & non-magnetic media, the tensor is real and symmetric. It can be shown that

$$\varepsilon_{ij} = \varepsilon_{ji} \tag{1.211}$$

└ the element values depend on orientation (choice) of the lab coordinate system with respect to that of the crystallographic system

We can orient three mutually orthogonal axes such that off-diagonal elements vanish…

$$\boldsymbol{\varepsilon} = \varepsilon_0 \begin{bmatrix} n_x^2 & 0 & 0 \\ 0 & n_y^2 & 0 \\ 0 & 0 & n_z^2 \end{bmatrix} = \begin{bmatrix} \varepsilon_x & 0 & 0 \\ 0 & \varepsilon_y & 0 \\ 0 & 0 & \varepsilon_z \end{bmatrix} \qquad (1.212)$$

principal indices *principal dielectric constants*

→ These are the *principal dielectric axes* of the crystal.

NB. A plane wave propagating in the *z*-direction can have two phase velocities, depending on its state of polarization

→ *x*-polarized light: $v_x = c/n_x$

y-polarized light: $v_y = c/n_y$

Optical Symmetry	Crystallographic System	Dielectric Tensor: *Optical system of axes*	Dielectric Tensor: *Crystallographic system of axes*
Isotropic	Cubic	$\varepsilon = \varepsilon_0 \begin{bmatrix} n^2 & 0 & 0 \\ 0 & n^2 & 0 \\ 0 & 0 & n^2 \end{bmatrix}$	$\varepsilon = \varepsilon_0 \begin{bmatrix} n^2 & 0 & 0 \\ 0 & n^2 & 0 \\ 0 & 0 & n^2 \end{bmatrix}$
Uniaxial	Hexagonal Tetragonal Trigonal	$\varepsilon = \varepsilon_0 \begin{bmatrix} n_o^2 & 0 & 0 \\ 0 & n_o^2 & 0 \\ 0 & 0 & n_e^2 \end{bmatrix}$	$\varepsilon = \varepsilon_0 \begin{bmatrix} n_o^2 & 0 & 0 \\ 0 & n_o^2 & 0 \\ 0 & 0 & n_e^2 \end{bmatrix}$
Biaxial	Orthorhombic	$\varepsilon = \varepsilon_0 \begin{bmatrix} n_x^2 & 0 & 0 \\ 0 & n_y^2 & 0 \\ 0 & 0 & n_z^2 \end{bmatrix}$	$\varepsilon = \varepsilon_0 \begin{bmatrix} n_x^2 & 0 & 0 \\ 0 & n_y^2 & 0 \\ 0 & 0 & n_z^2 \end{bmatrix}$
	Monoclinic	$\varepsilon = \varepsilon_0 \begin{bmatrix} n_x^2 & 0 & 0 \\ 0 & n_y^2 & 0 \\ 0 & 0 & n_z^2 \end{bmatrix}$	$\varepsilon = \varepsilon_0 \begin{bmatrix} \varepsilon_{11} & \varepsilon_{12} & 0 \\ \varepsilon_{21} & \varepsilon_{22} & 0 \\ 0 & 0 & \varepsilon_{33} \end{bmatrix}$
	triclinic	$\varepsilon = \varepsilon_0 \begin{bmatrix} n_x^2 & 0 & 0 \\ 0 & n_y^2 & 0 \\ 0 & 0 & n_z^2 \end{bmatrix}$	$\varepsilon = \varepsilon_0 \begin{bmatrix} \varepsilon_{11} & \varepsilon_{12} & \varepsilon_{13} \\ \varepsilon_{21} & \varepsilon_{22} & \varepsilon_{23} \\ \varepsilon_{31} & \varepsilon_{32} & \varepsilon_{33} \end{bmatrix}$

Table 1.2: Optical Symmetries and the Dielectric Tensor for Various Crystallographic Systems

Generalize…

Two normal modes of polarization for each direction of propagation

NB. $\boldsymbol{\varepsilon} = \boldsymbol{\varepsilon}(\omega)$

→ dispersion

Plane Waves in Homogeneous Media – the Normal Surface

*Assume plane wave propagation along a general direction…

$$\mathcal{E}(\mathbf{r},t) = \mathbf{E}\, e^{i(\omega t - \mathbf{k}\cdot\mathbf{r})} \qquad (1.213a)$$

$$\mathcal{H}(\mathbf{r},t) = \mathbf{H} e^{i(\omega t - \mathbf{k}\cdot\mathbf{r})} \qquad (1.213a)$$

…described by the wave vector

$$\mathbf{k} = \hat{\mathbf{s}}\left(\omega/c\right) n \qquad (1.214)$$

n: *effective refractive index*

$\hat{\mathbf{s}}$: a unit vector in the direction of "phase advance"
aka, informally: "propagation direction"

Using (1.210) & and (1.213) in Maxwell's curl equations (1.3) & (1.4) gives…

$$\mathbf{k}\times\mathbf{E} = \omega\mu\mathbf{H} \qquad (1.215)$$

$$\mathbf{k}\times\mathbf{H} = -\omega\boldsymbol{\varepsilon}\,\mathbf{H} \qquad (1.216)$$

(1.215) → (1.216) yields…

$$\mathbf{k}\times(\mathbf{k}\times\mathbf{E}) + \omega^2\mu\,\boldsymbol{\varepsilon}\,\mathbf{E} = 0 \qquad (1.217)$$

…or:

$$\mathbf{k}(\mathbf{k}\cdot\mathbf{E}) - k^2 E + \omega^2\mu\,\boldsymbol{\varepsilon}\,\mathbf{E} = 0 \qquad (1.218)$$

Using (1.214), (1.218) may be written in terms of the direction of propagation as…

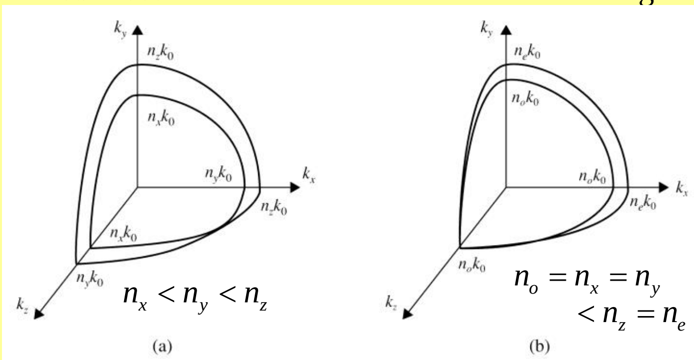
$$\hat{\mathbf{s}}(\hat{\mathbf{s}}\cdot\mathbf{E}) - \mathbf{E} + \left(\frac{\boldsymbol{\varepsilon}_r}{n^2}\right)\mathbf{E} = 0 \qquad (1.219)$$

(1.218) & (1.219) are equivalent wave equations from which the *eigenvalues* & *eigenvectors* may be extracted.

In the principal coordinate system, (1.212) → (1.218) yields a wave equation in matrix form…

$$\begin{bmatrix} \omega^2\mu\varepsilon_x - k_y^2 - k_z^2 & k_x k_y & k_x k_z \\ k_x k_y & \omega^2\mu\varepsilon_y - k_x^2 - k_z^2 & k_y k_z \\ k_x k_z & k_y k_z & \omega^2\mu\varepsilon_z - k_x^2 - k_y^2 \end{bmatrix} \begin{bmatrix} E_x \\ E_y \\ E_z \end{bmatrix} = 0 \qquad (1.220)$$

There are "nontrivial solutions" to (1.220) if its determinant vanishes → allowed frequencies ω for a given **k** → surfaces of allowed wavevector moduli for a given ω...



"normal surface"

Fig. 1.12: Octants of normal surfaces in momentum space (Y&Y, p. 32)

Reducing det [RHS (1.220)] = 0, one finds…

$$\left(\omega^2 n_x^2 / c^2 - k^2\right)\left(\omega^2 n_y^2 / c^2 - k^2\right)\left(\omega^2 n_z^2 / c^2 - k^2\right)$$

$$+k_x^2\left(\omega^2 n_y^2 / c^2 - k^2\right)\left(\omega^2 n_y^2 / c^2 - k^2\right) + k_y^2\left(\omega^2 n_x^2 / c^2 - k^2\right)\left(\omega^2 n_z^2 / c^2 - k^2\right)$$

$$+k_z^2\left(\omega^2 n_x^2 / c^2 - k^2\right)\left(\omega^2 n_y^2 / c^2 - k^2\right) = 0 \qquad (1.221)$$

(1.221) describes a surface in momentum space consisting of two shells that intersect in "*degenerate directions*"

"*optical axes*"

In general, the shells have four points in common

→ two lines go through the points & the origin

∴ Given a direction of propagation **s**…

→ two k values – the intersections of **s** & the normal surface

→ two different phase velocities ω/k

i.e., two different effective refractive indices
aka: eigenvalues of the index of refraction

The normal surface may equivalently be described via the vanishing of the determinant of the homogeneous part of the matrix equation from (1.219), whence (see Problem 1.28, text)

$$\left(n_x^2-n^2\right)\left(n_y^2-n^2\right)\left(n_z^2-n^2\right)+n^2[s_x^2\left(n_y^2-n^2\right)\left(n_z^2-n^2\right) \\ +s_y^2\left(n_x^2-n^2\right)\left(n_z^2-n^2\right)+s_z^2\left(n_x^2-n^2\right)\left(n_y^2-n^2\right)]=0 \qquad (1.222)$$

→ a quadratic equation in n^2

When $n \neq n_i$ ($i = x, y, z$), then (1.222) can be expressed as

$$\frac{s_x^2}{n^2-n_x^2}+\frac{s_y^2}{n^2-n_y^2}+\frac{s_z^2}{n^2-n_z^2}=\frac{1}{n^2} \qquad (1.223)$$

↳ "Fresnel's equation of wave normal"

(1.223) → two values of n^2 can be found for each (s_x, s_y, s_z)

*The eigenvectors associated with these eigenvalues can be found from (1.220), giving polarizations described by (see Problem 1.27, text)…

$$\mathbf{E}=\left(\frac{k_x}{k^2-\omega^2\mu\varepsilon_x},\frac{k_y}{k^2-\omega^2\mu\varepsilon_y},\frac{k_z}{k^2-\omega^2\mu\varepsilon_z}\right) \qquad (1.224)$$

Alternatively, (1.224) may be expressed by developing the matrix form of (1.219), as

$$\mathbf{E} = \left(\frac{s_x}{n^2 - n_x^2}, \frac{s_y}{n^2 - n_y^2}, \frac{s_z}{n^2 - n_z^2} \right) \qquad (1.225)$$

NB. In a non-absorbing medium, all the components of (1.224) – or (1.225) – are real.

→ the normal modes are linearly polarized

* Two linearly polarized normal modes $\mathbf{E_1}$ & $\mathbf{E_2}$ (with electric displacements $\mathbf{D_1}$ & $\mathbf{D_2}$) associated with n_1^2 & n_2^2

Since $\mathbf{D}_1 \cdot \mathbf{D}_2 = 0, \quad \mathbf{D}_i \perp \hat{\mathbf{s}} \quad (i = 1, 2)$

$\Rightarrow \mathbf{D}_1, \mathbf{D}_2 \,\&\, \hat{\mathbf{s}}$ are an orthogonal triad

(1.215) & (1.216) $\Rightarrow \mathbf{D} \,\&\, \mathbf{H} \perp \hat{\mathbf{s}} \quad \therefore \dfrac{\mathbf{E}\times\mathbf{H}}{|\mathbf{E}\times\mathbf{H}|} \neq \mathbf{s}$

i.e. The direction of energy flow may not be the same as the direction of phase advance.

Orthogonality of Normal Modes

NB. $\mathbf{D}, \mathbf{E} \,\&\, \mathbf{k} \perp \mathbf{H} \Rightarrow \mathbf{D}, \mathbf{E} \,\&\, \mathbf{k}$ lie in the same plane

It may be shown that these vectors also satisfy (Problem 1.27, text)

$$\mathbf{D}_1 \cdot \mathbf{D}_2 = 0 \qquad (1.226a)$$

$$\mathbf{D}_1 \cdot \mathbf{E}_2 = 0 \qquad (1.226b)$$

$$\mathbf{D}_2 \cdot \mathbf{E}_1 = 0 \qquad (1.226c)$$

$$\hat{\mathbf{s}} \cdot \mathbf{D}_1 = \hat{\mathbf{s}} \cdot \mathbf{D}_2 = 0 \qquad (1.226d)$$

NB. $\mathbf{E}_1$ & $\mathbf{E}_2$ are not, in general, orthogonal. The general orthogonality relation may be shown to be

$$\hat{\mathbf{s}} \cdot (\mathbf{E}_1 \times \mathbf{H}_2) = 0 \qquad (1.227)$$

→ power flow (in a loss-less anisotropic medium) along the direction of propagation is the sum of the power in each mode individually.

Classification of Media

- Information about propagation is contained in the

"normal surface"

$\hookrightarrow$ uniquely determined by n_x, n_y, n_z,

principal indices of refraction

Isotropic Case

For $n_x = n_z \equiv n_o$, (1.221) becomes…

$$\left[\left(\frac{\omega n_o}{c}\right)^2 - k^2\right]^2 \left(\frac{\omega n_o}{c}\right)^2 = 0 \qquad (1.228a)$$

→ normal surface is a degenerate sphere

Alternatively, (1.222) becomes…

$$\left(n_o^2 - n^2\right)^2 n_o^{\,2} = 0 \qquad (1.228b)$$

NB. The eigen-polarization is arbitrary.

Uniaxial Case

For $n_x = n_y = n_o$ and $n_z = n_e \neq n_o$ …

ordinary index *extraordinary index*

$n_o < n_e$ – positive medium
$n_o > n_e$ – negative medium

…(1.222) becomes

$$\left(n_o^2 - n^2\right)\left[\frac{1}{n^2} - \left(\frac{s_x^2 + s_y^2}{n_e^2}\right) - \frac{s_z^2}{n_o^2}\right] = 0 \qquad (1.229)$$

Normal surface *Sphere* + *Ellipsoid*

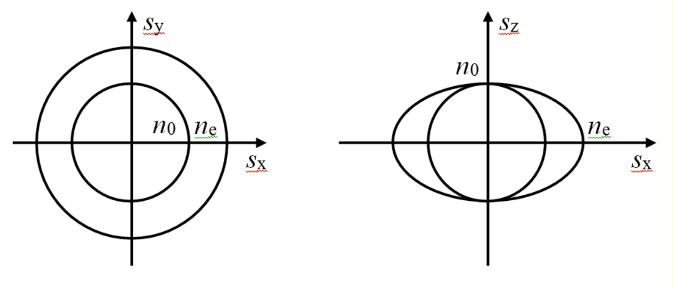


Figure 1.13: Sections of the normal surface for a positive uniaxial crystal.

Degenerate at two points, along the *z*-direction → *optical axis*

Biaxial Case

For $n_x < n_y < n_z$, we have the general case, as originally derived.

Sections of the normal surface can be extracted from (1.222)…

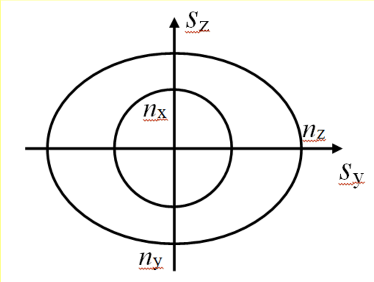


Figure 1.14: $S_x = 0$ section of the normal surface for a biaxial crystal.

$$s_x = 0: \quad \underbrace{\left[\frac{1}{n^2} - \left(\frac{s_y^2 + s_z^2}{n_x^2}\right)\right]}_{\text{circle}}\underbrace{\left[\frac{1}{n^2} - \left(\frac{s_z^2}{n_y^2} + \frac{s_y^2}{n_z^2}\right)\right]}_{\text{ellipse}} = 0 \qquad (1.230)$$

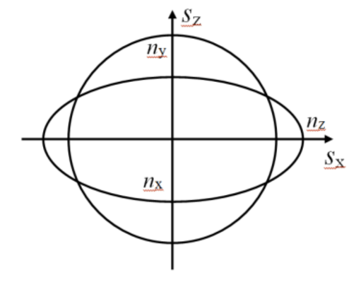


Figure 1.15: $S_y = 0$ section of the normal surface for a biaxial crystal.

$$s_y = 0: \quad \underbrace{\left[\frac{1}{n^2} - \left(\frac{s_x^2 + s_z^2}{n_y^2}\right)\right]}_{\text{circle}} \underbrace{\left[\frac{1}{n^2} - \left(\frac{s_x^2}{n_z^2} + \frac{s_z^2}{n_x^2}\right)\right]}_{\text{ellipse}} = 0 \qquad (1.231)$$

Degenerate at four points, in the *xy*-plane

→ *two optical axes*

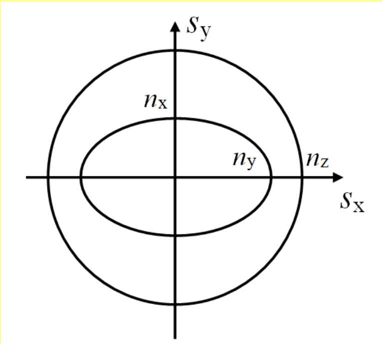


Figure 1.16: $S_z = 0$ section of the normal surface for a biaxial crystal.

$$s_z = 0: \quad \underbrace{\left[\frac{1}{n^2} - \left(\frac{s_x^2 + s_y^2}{n_z^2}\right)\right]}_{\text{circle}} \underbrace{\left[\frac{1}{n^2} - \left(\frac{s_x^2}{n_y^2} + \frac{s_y^2}{n_x^2}\right)\right]}_{\text{ellipse}} = 0 \qquad (1.232)$$