1.4 Pulse Propagation in Dispersive Media

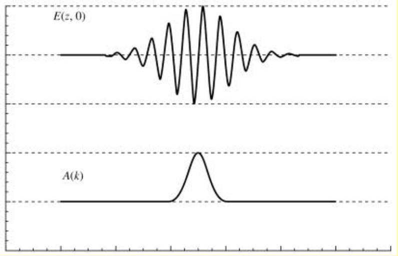
$\hookrightarrow$ Finite pulse duration

$\rightarrow$ Finite spread of frequencies

Propagation: superposition of plane waves

If we consider a linearly polarized field in the absence of absorption, then

$$\mathcal{E}(z,t) = \int_{-\infty}^{\infty} A(\beta) e^{i[\omega(\beta)t - \beta z]} d\beta \qquad (1.149)$$



$A(\beta)$ is the Fourier transform of $E(z,0)$

$\rightarrow |A(\beta)|^2$ is the power spectrum

Fig. 1.7: An optical pulse of finite extent and its Fourier spectrum in wavenumber (k) space.
Yariv & Yeh, 6th ed, p. 14

Chromatic dispersion: $v_p = v_p(\omega)$ $\rightarrow$ pulse distortion

Expand the dispersion relation about the central wave number β_0:

$$\omega(\beta) = \omega_0 + \left(\frac{d\omega}{d\beta}\right)_{\omega_0} (\beta - \beta_0) + \ldots \qquad (1.150)$$

(1.150) $\rightarrow$ (1.149) yields, to 1st order

$$\mathcal{E}(z,t) = e^{i(\omega_0 t - \beta_0 z)} \int_{-\infty}^{\infty} A(\beta) \exp\left\{ i \left[\left(\left.\frac{d\omega}{d\beta}\right|_{\omega_0} \right) t - z \right] (\beta - \beta_0) \right\} d\beta \quad (1.151)$$

Let $$V(\xi) = \int_{-\infty}^{\infty} A(\beta) e^{i\xi(\beta - \beta_0)} d\beta \qquad (1.152)$$

where $$\xi = \left(\left.\frac{d\omega}{d\beta}\right|_{\omega_0} \right) t - z \qquad (1.153)$$

(1.152) $\rightarrow$ (1.151) yields

$$\mathcal{E}(z,t) = e^{i(\omega_0 t - \beta_0 z)} V(\xi) \qquad (1.154)$$

Envelope function

Thus, to 1st order, the peak of the envelope may be determined from

$$\frac{dV}{dt} = \frac{dV}{d\xi}\frac{d\xi}{dt} = 0 \Rightarrow \frac{d\xi}{dt} = 0 \qquad (1.155)$$

(1.153) & (1.155) yield

$$v_g = \frac{dz}{dt} = \left.\frac{d\omega}{d\beta}\right|_{\omega_0} \qquad (1.156)$$

group velocity

(1.154) shows that, apart from an overall phase factor, to 1st order the pulse travels undistorted in shape with an envelope velocity given by (1.156)

$\Rightarrow v_g \neq v_p$ in general, where $v_g = v_e$ is the energy velocity.

Recall (1.97) and (1.100):

$$\beta = \frac{n(\omega)\omega}{c} \qquad (1.157)$$

The phase velocity [see (1.82)] is

$$v_p = \frac{\omega}{\beta} = \frac{c}{n(\omega)} \qquad (1.158)$$

Applying (1.156) to (1.157):

$$v_g = \left(\frac{d\beta}{d\omega}\right)^{-1} = \frac{c}{n(\omega) + \omega \, dn/d\omega} \qquad (1.159)$$

Or, in terms of the free space (i.e., $\omega=2\pi c/\lambda$) wavelength λ:

$$v_g = c/N \qquad (1.160)$$

where

$$N = n\lambda - \lambda\frac{dn}{d\lambda} \qquad (1.161)$$

is the group index.

Group Velocity Dispersion

If (1.150) to 1st order is insufficient

→ very temporally narrow pulses → broad $A(\beta)$

must go to 2nd order:

$$\frac{1}{2}\left(\frac{d^2\omega}{d\beta^2}\right)_{\omega_0}(\beta-\beta_0)^2 \qquad (1.162)$$

As the pulse travels, it distorts…

…"pulse spreading"

Two identical pulses, having a difference in central wave number, $\Delta\beta$, will differ in group velocity as

$$\Delta v_g = \left(\frac{d}{d\beta}v_g\right)\Delta\beta = \left(\frac{d^2\omega}{d\beta^2}\right)_{\omega_0}\Delta\beta \qquad (1.163)$$

If a pulse travels for a time T across a distance L, the spread in positions will be $(\Delta v_g)T$.

To more precisely quantify this, we define

$$D = \frac{1}{L}\frac{dT}{d\lambda} \qquad (1.164)$$

GVD – group velocity dispersion

"the pulse spreading per unit bandwidth per unit length"

Since: $T = L/v_g$ (1.165)

(1.160) → (1.165), using (1.161), yields

$$T = \frac{L}{c}\left(n-\lambda\frac{dn}{d\lambda}\right) \qquad (1.166)$$

(1.166) → (1.164) yields $D = -\frac{\lambda}{c}\frac{d^2n}{d\lambda^2}$ (1.167)

1.5 Polarization States & Representations

Orientation of the electric field in space and time…

Illustrate – monochromatic plane wave in an isotropic & homogeneous medium

a *transverse wave* $\therefore \mathbf{k}\cdot\boldsymbol{\mathcal{E}} = 0$, see (1.57)

NB. $\mathbf{k} = \dfrac{n\omega}{c}$ – see (1.48), (1.86) – may be complex

two independent directions of oscillation

Unpolarized… if directions are uncorrelated

resultant direction is random

From (1.47), we recall: $\boldsymbol{\mathcal{E}}(\mathbf{r},t) = \mathbf{A}e^{i(\omega t - \mathbf{k}\cdot\mathbf{r})}$

specific polarization state

Consider propagation along the *z*-axis. The independent field components are…

$$\mathcal{E}_x = A_x \cos(\omega t - kz + \delta_x) \qquad (1.168a)$$

$$\mathcal{E}_y = A_y \cos(\omega t - kz + \delta_y) \qquad (1.168b)$$

…independent phases: $-\pi < \delta_{x,y} \le \pi$

We superimpose these two orthogonal oscillations. WOLOG, we set $z = 0$, hence (1.168a & b) become…

$$\mathcal{E}_x = A_x \cos(\omega t + \delta_x) \qquad (1.169a)$$

$$\mathcal{E}_y = A_y \cos(\omega t + \delta_y) \qquad (1.169b)$$

We define the relative phase: $\delta = \delta_y - \delta_x$ (1.170)

$-\pi < \delta \le \pi$

…and consider first the two special cases.

Linear Polarization States

→ the electric field vector vibrates in a fixed direction

$$\delta = 0 \quad \text{or} \quad \delta = \pi \qquad (1.171)$$

$\delta = 0$ → in phase; $\delta = \pi$ → out of phase

(1.169) & (1.171) yield:

$$\frac{\mathcal{E}_y}{\mathcal{E}_x} = \frac{A_y}{A_x} \quad \text{or} \quad -\frac{A_y}{A_x} \qquad (1.172)$$

defines a plane → plane polarized light

Since the amplitudes are independent, the field vector can be oriented along any direction in the *xy*-plane

Circular Polarization States

→ the electric field vector rotates uniformly

$$\Rightarrow A_x = A_y \quad \text{and} \quad \delta = +\frac{\pi}{2} \quad \text{or} \quad -\frac{\pi}{2} \qquad (1.173)$$

LHCP (CW rotation in the *xy* plane) ← $+\frac{\pi}{2}$

RHCP (CCW rotation of the field vector) ← $-\frac{\pi}{2}$

NB. This definition is consistent with a RHCP photon having positive angular momentum.

Elliptic Polarization State – most general case

→ The electric field vector traces out an ellipse

Using the parametric equations (1.168), and various trig identities, one may derive (see Problem 1.9, text)

$$\left(\frac{\mathcal{E}_x}{A_x}\right)^2 + \left(\frac{\mathcal{E}_y}{A_y}\right)^2 - \frac{2\cos\delta}{A_x A_y}\mathcal{E}_x\mathcal{E}_y = \sin^2\delta \qquad (1.174)$$

This is a conic section (not the most general)…
an ellipse confined to a rectangle of $2A_x$ by $2A_y$

NB. The principal axes of the ellipse are not typically aligned with the reference coordinate system; thus we also need to describe the orientation of the ellipse.

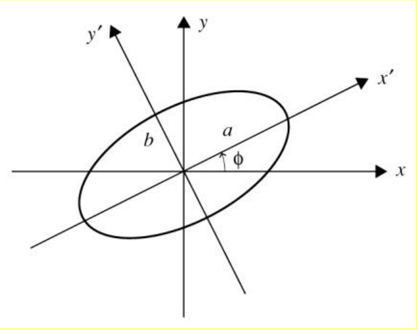


Fig. 1.8: The polarization ellipse

(1.174) may be diagonalized, by a coordinate system rotation, to yield…

$$\left(\frac{\mathcal{E}_{x'}}{a}\right)^2 + \left(\frac{\mathcal{E}_{y'}}{b}\right)^2 = 1 \qquad (1.175)$$

… where a & b are the semi major and minor principal axes; $\mathcal{E}_{x'}$, $\mathcal{E}_{y'}$, are the field components in the new coordinate system.

The lengths of the semi major & minor axes may be found as (Problem 1.10, text)

$$a^2 = A_x{}^2 \cos^2\phi + A_y{}^2 \sin^2\phi + 2A_xA_y \cos\delta \cos\phi \sin\phi \qquad (1.176a)$$

$$b^2 = A_x{}^2 \sin^2\phi + A_y{}^2 \cos^2\phi - 2A_xA_y \cos\delta \cos\phi \sin\phi \qquad (1.176b)$$

As seen in Fig. 1.8, the inclination angle, between the x & x' axes, is ϕ. It may be expressed as (Problem 1.10, text)

$$\tan(2\phi) = \left(\frac{2A_xA_y}{A_x{}^2 - A_y{}^2}\right)\cos\delta \qquad (1.177)$$

NB. If ϕ is a solution, $\phi + n\pi/2$ is also a solution ($n \in$ Z).

*We define the ellipticity of the polarization ellipse as

$$e = \pm b/a \qquad (1.178)$$

where + is RHP, and – is LHP

The ellipticity angle is $\theta \equiv \arctan e$ (1.179)

NB. $\sin\delta \begin{cases} >0 \\ <0 \end{cases}$ is $\begin{cases} CW \\ CCW \end{cases}$ vector rotation $\Rightarrow \begin{cases} LHP \\ RHP \end{cases}$

- see Problem 1.11, text

NB. An elliptical polarization state can always be decomposed into two orthogonal components, their relative phase shift between –π and +π. However, in the principal coordinate system, the relative phase shift is ± π/2, depending on handedness.

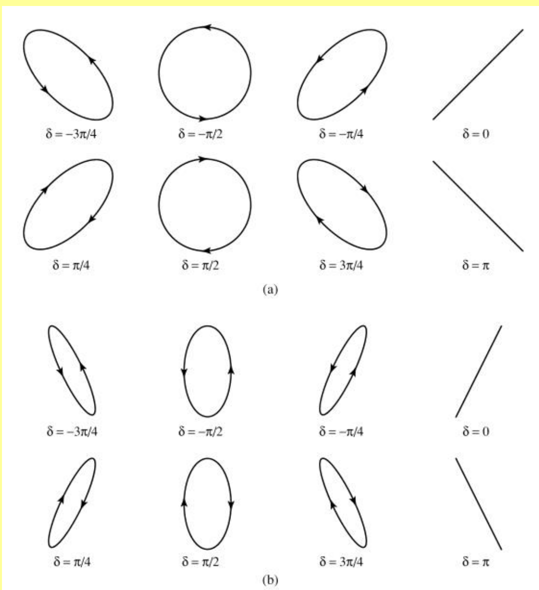


Fig 1.9: Polarization ellipses at various phase angles;

(a) $\mathcal{E}_x = \cos(\omega t - kz)$

$\mathcal{E}_y = \cos(\omega t - kz + \delta)$

(b) $\mathcal{E}_x = \frac{1}{2}\cos(\omega t - kz)$

$\mathcal{E}_y = \cos(\omega t - kz + \delta)$

Complex Number Representation

Recall (1.168) – the polarization state is encoded in the complex amplitude **A** of the plane wave. We may thus define

$$\chi = e^{i(\delta_y - \delta_x)} \left(A_y \big/ A_x \right) \equiv e^{i\delta} \tan\psi \qquad (1.180)$$

fully describes the polarization state

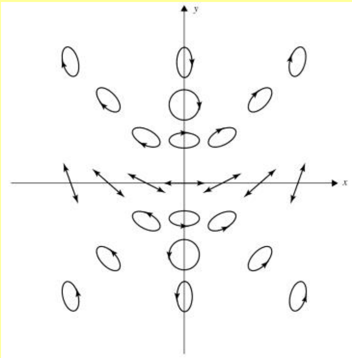


Fig. 1.10: Complex number representation of polarization states (PS)

Each point is a unique polarization state

x-axis: linear PS

(1,0): ψ = 45˚, LPS

(-1,0) : ψ = 45˚, LPS   !? → different δ

(0,1): LHCP

(0,-1): RHCP

NB. $0 \leq \psi \leq \pi/2$ describes the azimuth angle with respect to the real axis.

*The inclination angle can be described in terms of δ & ψ from (1.177) & (1.180) as (see Problem 1.13, text)

$$\tan(2\phi) = \frac{2\,\mathrm{Re}\{\chi\}}{1 - |\chi|^2} = \cos\delta \tan(2\psi) \qquad (1.181)$$

*Describing the ellipticity angle in terms of δ & ψ is slightly more involved (see Problem 1.13, text)

$$\sin(2\theta) = \frac{-2\,\mathrm{Im}\{\chi\}}{1 + |\chi|^2} = -\sin(2\psi)\sin\delta \qquad (1.182)$$

Jones Vector Representation

- Used to express the polarization state of a plane wave

- Used in the Jones calculus, where propagation of plane waves through optical systems are described

The plane wave of (1.168) is expressed as

$$\mathbf{J} = \begin{bmatrix} A_x e^{i\delta_x} \\ A_y e^{i\delta_y} \end{bmatrix} \qquad (1.183)$$

a complex vector

→ specifies the polarization state uniquely

NB. The electric field vectors are found via $\mathcal{E}_{x,y} = \mathrm{Re}\{J_{x,y} e^{i\omega t}\}$

NB. Normalization of the Jones vector discards absolute phase information, retaining only polarization information. The normalized Jones vector satisfies

$$\mathbf{J}^* \cdot \mathbf{J} = 1 \qquad (1.184)$$

***For linearly polarized light**

$$\mathbf{J} = \begin{bmatrix} \cos\psi \\ \sin\psi \end{bmatrix} \qquad (1.185)$$

NB. The linear polarization state orthogonal to this is obtained by shifting $\psi \to \psi + \pi/2$ in (1.185)

$$\mathbf{J} = \begin{bmatrix} -\sin\psi \\ \cos\psi \end{bmatrix} \qquad (1.186)$$

NB. The special case of $\psi = 0$ represents *LP* field vectors oriented along the coordinate axes as

$$\hat{\mathbf{x}} = \begin{bmatrix} 1 \\ 0 \end{bmatrix} \qquad \hat{\mathbf{y}} = \begin{bmatrix} 0 \\ 1 \end{bmatrix} \qquad (1.187)$$

***For circularly polarized light**

$$\mathbf{R} = \frac{1}{\sqrt{2}}\begin{bmatrix} 1 \\ -i \end{bmatrix}, \text{ RCP} \qquad (1.188)$$

$$\mathbf{L} = \frac{1}{\sqrt{2}}\begin{bmatrix} 1 \\ i \end{bmatrix}, \text{ LCP} \qquad (1.189)$$

NB. The two circular polarization states are orthogonal:

$$\mathbf{R}^* \cdot \mathbf{L} = 0 \qquad (1.190)$$

*Any pair of orthogonal Jones vectors can be used as a basis of the space spanned by all Jones vectors.

e.g. Any polarization can be represented by $\hat{\mathbf{x}}$ & $\hat{\mathbf{y}}$ or $\mathbf{R}$ & $\mathbf{L}$

These can also be represented by other bases…

e.g. circular by linear…

$$\mathbf{R} = \frac{1}{\sqrt{2}}\left(\hat{\mathbf{x}} - i\,\hat{\mathbf{y}}\right) \qquad (1.191)$$

$$\mathbf{L} = \frac{1}{\sqrt{2}}\left(\hat{\mathbf{x}} + i\,\hat{\mathbf{y}}\right) \qquad (1.192)$$

e.g. linear by circular…

$$\hat{\mathbf{x}} = \frac{1}{\sqrt{2}}\left(\mathbf{R} + \mathbf{L}\right) \qquad (1.193)$$

$$\hat{\mathbf{y}} = \frac{i}{\sqrt{2}}\left(\mathbf{R} - \mathbf{L}\right) \qquad (1.194)$$

***For elliptically polarized light**

$$\mathbf{J}(\psi,\delta) = \begin{bmatrix} \cos\psi \\ e^{i\delta}\sin\psi \end{bmatrix} \qquad (1.195)$$

NB. (1.195) is the Jones vector representation of (1.180), which may be expressed as

$$\chi = {}^{J_y}\!/_{J_x} \qquad (1.196)$$

Polarization Ellipse	Jones Vector	(δ, ψ)	(ϕ, θ)	Stokes Vector
	$\begin{bmatrix}1\\0\end{bmatrix}$	$(0, 0)$	$(0, 0)$	$\begin{bmatrix}1\\1\\0\\0\end{bmatrix}$
	$\begin{bmatrix}0\\1\end{bmatrix}$	$(0, \pi/2)$	$(\pi/2, 0)$	$\begin{bmatrix}1\\-1\\0\\0\end{bmatrix}$
	$\frac{1}{\sqrt{2}}\begin{bmatrix}1\\1\end{bmatrix}$	$(0, \pi/4)$	$(\pi/4, 0)$	$\begin{bmatrix}1\\0\\1\\0\end{bmatrix}$
	$\frac{1}{\sqrt{2}}\begin{bmatrix}1\\-1\end{bmatrix}$	$(\pi, \pi/4)$	$(-\pi/4, 0)$	$\begin{bmatrix}1\\0\\-1\\0\end{bmatrix}$
	$\frac{1}{\sqrt{2}}\begin{bmatrix}1\\-i\end{bmatrix}$	$(-\pi/2, \pi/4)$	$(0, \pi/4)$	$\begin{bmatrix}1\\0\\0\\-1\end{bmatrix}$
	$\frac{1}{\sqrt{2}}\begin{bmatrix}1\\i\end{bmatrix}$	$(\pi/2, \pi/4)$	$(0, -\pi/4)$	$\begin{bmatrix}1\\0\\0\\1\end{bmatrix}$

Table 1.1: Various Representations of Polarization States

Yariv & Yeh, p. 27.

Stokes Parameters & Partially Polarized Light

A monochromatic plane wave is inherently polarized.

BUT: For light that is NOT monochromatic – i.e. is polychromatic – a time variation can exist between the amplitudes, and phases, of the field components.

→ a polarization ellipse that fluctuates in time.

Measured state of polarization depends on speed of observation τ_D

time averaging

*Consider "quasi-monochromatic" ($\Delta\omega << \omega$) waves. Thee are still described by (1.168), but now

$$\boldsymbol{\mathcal{E}}(\mathbf{r},t) = \boldsymbol{\mathcal{A}}(t)\, e^{i(\omega t - \mathbf{k}\cdot\mathbf{r})} \qquad (1.197)$$

slowly varying complex amplitude

centre frequency

i.e. $\mathcal{A}(t) \simeq$ constant over $1/\Delta\omega$

NB. $\tau_D < 1/\Delta\omega$, else $\mathcal{A}(t)$ may change significantly

Their polarization state can be described by the following time averages:

$$S_0 = \left\langle \left\langle \mathcal{A}_x^2 + \mathcal{A}_y^2 \right\rangle \right\rangle \qquad (1.198a)$$

$$S_1 = \left\langle \left\langle \mathcal{A}_x^2 - \mathcal{A}_y^2 \right\rangle \right\rangle \qquad (1.198b)$$

$$S_2 = 2\left\langle \left\langle \mathcal{A}_x \mathcal{A}_y \cos\delta \right\rangle \right\rangle \qquad (1.198c)$$

$$S_3 = 2\left\langle \left\langle \mathcal{A}_x \mathcal{A}_y \sin\delta \right\rangle \right\rangle \qquad (1.198d)$$

These are the *Stokes parameters* (dimension: intensity)

NB. $\mathcal{A}_x$, $\mathcal{A}_y$ & δ are time-dependent.

The Stokes parameters satisfy…

$$S_1^2 + S_2^2 + S_3^2 \le S_0^2 \qquad (1.199)$$

(equality only for polarized waves)

*Various Stokes parameters – vector representation

$$\mathbf{S} = \left(S_0, S_1, S_2, S_3\right) \qquad (1.200)$$

also as a column vector (see Table 1.1)

- Unpolarized light:

no preference between $\mathcal{A}_x$ & $\mathcal{A}_y$
δ is a random function of time

$$\Rightarrow \mathbf{S} = \left(1, 0, 0, 0\right) \qquad (1.200a)$$

- Horizontally polarized light: $\mathcal{A}_y = 0$, $\delta = 0$

$$\Rightarrow \mathbf{S} = (1,1,0,0) \qquad (1.200b)$$

- Vertically polarized light: $\mathcal{A}_x = 0$, $\delta = 0$

$$\Rightarrow \mathbf{S} = (1,-1,0,0) \qquad (1.200c)$$

- Right hand circularly polarized light: $\mathcal{A}_x = \mathcal{A}_y$, $\delta = -\pi/2$

$$\Rightarrow \mathbf{S} = (1,0,0,-1) \qquad (1.200d)$$

- Left hand circularly polarized light: $\mathcal{A}_x = \mathcal{A}_y$, $\delta = +\pi/2$

$$\Rightarrow \mathbf{S} = (1,0,0,1) \qquad (1.200d)$$

NB. $S_0 = 1$ (normalization); $-1 \le S_1, S_2, S_3 \le 1$
If all are 0 $\rightarrow$ unpolarized.
If $S_1^2 + S_2^2 + S_3^2 = 1$ $\rightarrow$ polarized

We may therefore define…

$$\gamma = \frac{\sqrt{S_1^2 + S_2^2 + S_3^2}}{S_0} \qquad (1.201)$$

degree of polarization

…where $0 \le \gamma \le 1$.

NB.

- S_1 describes the linear polarization along the *x* & *y* axes, with probabilities ½(1 + S_1) & ½ (1 – S_1), respectively.
- S_2 describes the linear polarization along direction $\phi = \pm 45^\circ$ to the *x*-axis, with probabilities ½(1 + S_2) & ½ (1 – S_2), respectively.
- S_3 describes the degree of circular polarization, LH & RH, with probabilities ½(1 + S_3) & ½ (1 – S_3), respectively.

*For polarized light, the relationships between Stokes parameters and ψ & δ are found via (1.180) as

$$S_0 = 1 \qquad (1.120a)$$

$$S_1 = \cos(2\psi) \qquad (1.202b)$$

$$S_2 = \sin(2\psi)\cos\delta \qquad (1.202c)$$

$$S_3 = \sin(2\psi)\sin\delta \qquad (1.202d)$$