1.2 Overview of General Material Systems

We have discussed some concepts under rather idealistic conditions

→ *linear, nondispersive, homogeneous, isotropic*

For such dielectric media

$$\mathcal{P} = \varepsilon_0 \chi \mathcal{E} \qquad (1.83)$$

electric susceptibility

From (1.1), using (1.83),

$$\varepsilon = \varepsilon_0 (1 + \chi) \qquad (1.84)$$

Linearity: $\mathcal{P}$ linearly related to $\mathcal{E}$

Nondispersive: Instantaneous response

i.e., $\mathcal{E}(t)$ determines $\mathcal{P}(t)$ only at time t

Homogeneous: χ does not depend on **r**

Isotropic: χ does not depend on the direction of $\mathcal{E}$

What happens if one of these properties is not satisfied?

Inhomogeneous Media

…*linear, nondispersive, isotropic*

NB. Unless otherwise specified, let us assume *nonmagnetic* media

Now, $\chi = \chi(\mathbf{r})$, $\varepsilon = \varepsilon(\mathbf{r})$. Recall (1.44) for the propagation of an EM wave in an isotropic medium, now written as

$$\nabla^2 \mathcal{E} - \mu_0 \varepsilon \frac{\partial^2 \mathcal{E}}{\partial t^2} = \nabla(\nabla \cdot \mathcal{E}) \qquad (1.85)$$

(1.5) asserts $\nabla \cdot \mathcal{D} = 0$ for source-free media, and (1.1) states $\mathcal{D} = \varepsilon \mathcal{E}$ so $\nabla \cdot \mathcal{E} \neq 0$ in general. Hence

$$\nabla \cdot \mathcal{E} = \nabla \cdot \left(\mathcal{D} / \varepsilon \right) = \varepsilon^{-1} \nabla \cdot \mathcal{D} + \mathcal{D} \cdot \nabla \left(1/\varepsilon \right) = -\mathcal{E} \cdot \left(\frac{\nabla \varepsilon}{\varepsilon} \right) \qquad (1.86)$$

(1.86) → (1.85) yields...

fractional change in ε

$$\nabla^2 \boldsymbol{\mathcal{E}} - \mu_0 \varepsilon \frac{\partial^2 \boldsymbol{\mathcal{E}}}{\partial t^2} = -\nabla\left(\varepsilon^{-1} \boldsymbol{\mathcal{E}} \cdot \nabla \varepsilon\right) \tag{1.87}$$

Clearly, RHS(1.87) may only be neglected if $\nabla\left(\varepsilon^{-1} \boldsymbol{\mathcal{E}} \cdot \nabla \varepsilon\right) \ll 1$ over a wavelength.

(1.87) may be interpreted as a homogeneous medium with a refractive index n that is perturbed by a spatially dependent Δn.

Recall (1.45):

$$\nabla^2 \boldsymbol{\mathcal{E}} - \mu_0 \varepsilon \frac{\partial^2 \boldsymbol{\mathcal{E}}}{\partial t^2} = 0$$

Perturbing n in $\mu_0\varepsilon = n^2/c^2$, we take

$$\left(n + \Delta n\right)^2 c^{-2} \approx \left(n^2 + 2n\Delta n\right) c^{-2} \tag{1.88}$$

Using (1.88), we can revise (1.45) as

$$\nabla^2 \boldsymbol{\mathcal{E}} - \mu_0 \varepsilon \frac{\partial^2 \boldsymbol{\mathcal{E}}}{\partial t^2} \approx 2\mu_0 \varepsilon_0 n \Delta n \frac{\partial^2 \boldsymbol{\mathcal{E}}}{\partial t^2} \tag{1.89}$$

We can therefore interpret (1.87) as a wave equation with a radiation source term

$$S = \mu_0 \frac{\partial^2 \left(\Delta \boldsymbol{\mathcal{P}}\right)}{\partial t^2} \tag{1.90}$$

created by a perturbation of a polarization

$$\Delta \boldsymbol{\mathcal{P}} = 2\varepsilon_0 n \Delta n \boldsymbol{\mathcal{E}} \tag{1.91}$$

Nonlinear Media

…homogeneous, isotropic

→ Wave equations (1.45) & (1.46) are no longer applicable.

→ must derive a nonlinear wave equation.

From (1.42):

$$\nabla \times \left(\nabla \times \boldsymbol{\mathcal{E}}\right) + \mu_0 \frac{\partial}{\partial t}\left(\nabla \times \boldsymbol{\mathcal{H}}\right) = 0 \tag{1.92}$$

Using (1.4), this becomes…

$$\nabla\times\left(\nabla\times\mathcal{E}\right)+\mu_0\frac{\partial^2\mathcal{D}}{\partial t^2}=0 \qquad (1.93)$$

Recall (1.1): $\mathcal{D}=\varepsilon_0\mathcal{E}+\mathcal{P}$

(1.1) into (1.93), using a vector identity on the double curl, gives

$$\nabla\left(\nabla\cdot\mathcal{E}\right)-\nabla^2\mathcal{E}+\mu_0\varepsilon_0\frac{\partial^2\mathcal{E}}{\partial t^2}+\mu_0\frac{\partial^2\mathcal{P}}{\partial t^2}=0 \qquad (1.94)$$

Using (1.5) for homogeneous material, (1.94) becomes

$$\nabla^2\mathcal{E}-\mu_0\varepsilon_0\frac{\partial^2\mathcal{E}}{\partial t^2}=\mu_0\frac{\partial^2\mathcal{P}}{\partial t^2} \qquad (1.95)$$

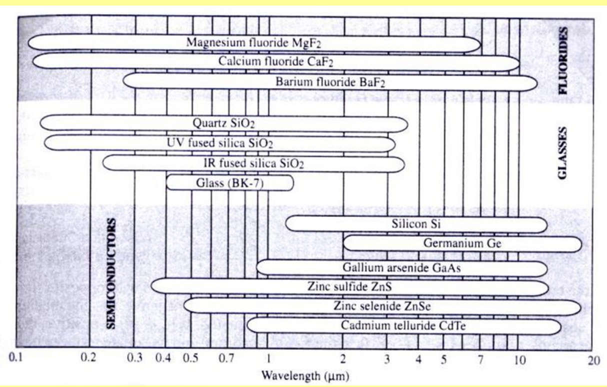
NB. For nondispersive media, $\mathcal{P}=\mathcal{F}(\mathcal{E})$ only.

What about dispersive media?

There is a relationship between absorption and dispersion.

We first discuss absorption and dispersion, before addressing the relationship.

Absorption



Optical materials are generally transparent only within certain spectral windows.

Figure 1.3: The spectral bands for which selected optical materials transmit light.

(From Saleh & Teich, 2nd ed, p. 171)

We will presume linear media and use a phenomenological approach
→ treat the susceptibility as a complex function

$$\chi=\chi'+i\chi'' \qquad (1.96)$$

Wave equations (1.45) and (1.46) are still valid, but (1.48) implies k must be complex because (1.96) → (1.84) means ε is complex:

$$k = \omega\sqrt{\mu\varepsilon} = k_0\sqrt{1+\chi} = k_0\sqrt{1+\chi' + i\chi''} \qquad (1.97)$$

NB. The free space wave number is

$$k_0 = \omega/c \qquad (1.98)$$

Let us define

$$k = \beta - i\alpha/2 \qquad (1.99)$$

absorption coefficient (> 0) — α

propagation constant — β

Why the 2? $\psi \sim Ae^{-ikz}$, see (1.47)

$\rightarrow Ae^{-\alpha z/2}e^{-i\beta z}$, from (1.99)

$\therefore \quad \left|\psi\psi^*\right| \sim e^{-\alpha z} \quad \leftarrow$ intensity

The effective refractive index n is defined as

$$\beta = nk_0 \qquad (1.100)$$

(1.100) → (1.99), using (1.54), (1.84), (1.97), and assuming a nonmagnetic medium, gives

$$n - i\alpha/2k_0 = \sqrt{\varepsilon/\varepsilon_0} = \sqrt{(1+\chi') + i\chi''} \qquad (1.101)$$

The square root has 2 complex solutions. The choice is made on the basis that $n > 0$, $\alpha > 0$, given that $\chi'' < 0$ for an absorbing medium.

The impedance also is now complex. From (1.61) and (1.84),

$$\eta = \sqrt{\mu_0/\varepsilon} = \frac{\eta_0}{\sqrt{1+\chi}} \qquad (1.102)$$

Let us consider two special cases.

Weakly Absorbing Media **:** $|\chi''| \ll |1+\chi'|$

Let us define δ such that

$$\sqrt{1+\chi'+i\chi''} = \sqrt{1+\chi'}\sqrt{1+i\delta} \qquad (1.103)$$

Thus

$$\delta = \frac{\chi''}{1+\chi'} \qquad (1.104)$$

Since $\delta << 1$, (1.103) → (1.101) approximates as

$$n - i\alpha/2k_0 \approx \sqrt{1+\chi'}\left(1+i\tfrac{1}{2}\delta\right) \qquad (1.105)$$

(1.105) yields

$$n \approx \sqrt{1+\chi'} \qquad (1.106)$$

and $\alpha/2k_0 \approx \frac{1}{2}\delta\sqrt{1+\chi'} \quad \Rightarrow \quad \alpha \approx -\frac{k_0}{n}\chi''$ (1.107)

Strongly Absorbing Media **:** $|\chi''| \gg |1+\chi'|$

From (1.101),

$$n - i\alpha/2k_0 \approx \sqrt{i\chi''} = \sqrt{-i}\sqrt{-\chi''} = \pm\frac{1}{\sqrt{2}}(1-i)\sqrt{-\chi''} \qquad (1.108)$$

Thus

$$n \approx \sqrt{-\tfrac{1}{2}\chi''} \qquad (1.109)$$

$$\alpha \approx k_0\sqrt{-2\chi''} = 2k_0 n \qquad (1.110)$$

Dispersive Media

...linear, homogeneous, isotropic

Now $\mathcal{E} \rightarrow \mathcal{P}$ … a dynamic relationship

↳ a time delay → system “memory”

Figure 1.4: Linear systems approach.

$$\mathcal{E}(t) \longrightarrow \boxed{\chi(t)} \longrightarrow \mathcal{P}(t)$$

$\mathcal{E}(t)$ induces a $\mathcal{P}(t)$ that is a superposition of all the effects of $\mathcal{E}(t')$ for all $t' \le t$

$$\mathcal{P}(t) = \varepsilon_0 \int_{-\infty}^{\infty} \chi(t-t')\mathcal{E}(t')\,dt' \qquad (1.111)$$

a convolution integral

$\varepsilon_0 \chi(t) \quad \leftarrow \quad$ impulse response function

We may also describe the medium by its transfer function…

$$\varepsilon_0 \chi(\omega) = \varepsilon_0 \int_{-\infty}^{\infty} \chi(t) e^{-i\omega t}\,dt \qquad (1.112)$$

frequency dependent susceptibility

… a Fourier transform

In a dispersive media, in the frequency domain, (1.83) implies

$$\mathbf{P} = \varepsilon_0 \chi(\omega)\mathbf{E} \qquad (1.113)$$

Likewise, (1.1) implies

$$\mathbf{D} = \varepsilon(\omega)\mathbf{E} \qquad (1.114)$$

And thus, via (1.1), (1.113) & (1.114) imply

$$\varepsilon(\omega) = \varepsilon_0\left[1 + \chi(\omega)\right] \qquad (1.115)$$

Also, via (1.48)

$$k = \omega\sqrt{\mu_0 \varepsilon(\omega)} \qquad (1.116)$$

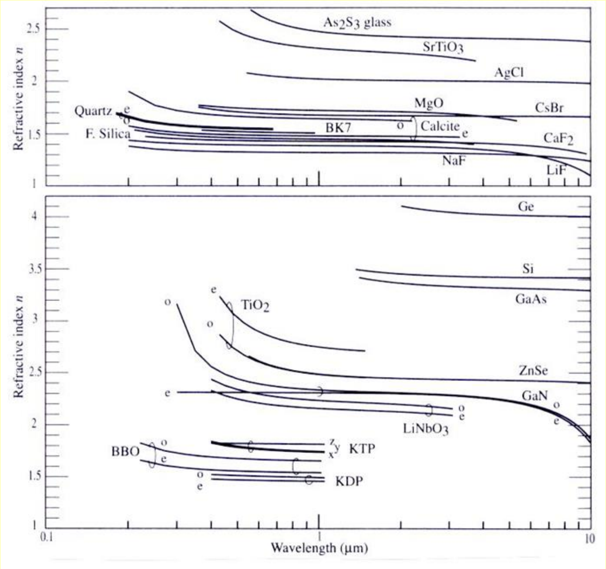


Figure 1.5: Wavelength dependence of the refractive index for selected optical materials, including glasses, crystals, and semi-conductors.

From Saleh & Teich, 2nd ed. p. 174.

1.3 The Kramers-Kroning Relations

A dispersive material MUST be absorptive.
The absorption coefficient MUST be wavelength dependent.

The relation between α and n is established through the real and imaginary parts of χ, or equivalently, through the real and imaginary parts of ε.

Let us consider the complex permittivity

$$\varepsilon(\omega) = \varepsilon'(\omega) - i\varepsilon''(\omega) \qquad (1.117)$$

where by (1.115) & (1.96)

$$\varepsilon'(\omega) = \varepsilon_0 n^2(\omega) = \varepsilon_0\left[1 + \chi'(\omega)\right] \qquad (1.118)$$

$$\varepsilon''(\omega) = -\varepsilon_0 \chi''(\omega) \qquad (1.119)$$

The Kramers-Kronig relations enable us to find $\varepsilon'(\omega)$ if we know $\varepsilon''(\omega)$, and vice-versa, at all frequencies.

Recall (1.114): $\mathbf{D}(\omega) = \varepsilon(\omega)\mathbf{E}(\omega)$

a temporally nonlocal connection between $\mathcal{E}(t)$ and $\mathcal{D}(t)$

We can reveal this through the Fourier transforms

$$\mathcal{D}(t) = \int_{-\infty}^{\infty} \mathbf{D}(\omega) e^{i\omega t} d\omega \qquad (1.120)$$

$$\mathcal{E}(t) = \int_{-\infty}^{\infty} \mathbf{E}(\omega) e^{i\omega t} d\omega \qquad (1.121)$$

(1.114) into (1.120) yields

$$\mathcal{D}(t) = \int_{-\infty}^{\infty} \varepsilon(\omega)\mathbf{E}(\omega) e^{i\omega t} d\omega \qquad (1.122)$$

The inverse Fourier transform of (1.121) is

$$\mathbf{E}(\omega) = \frac{1}{2\pi}\int_{-\infty}^{\infty} \mathcal{E}(t) e^{-i\omega t} dt \qquad (1.123)$$

(1.123) into (1.122) yields

$$\mathcal{D}(t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} \varepsilon(\omega) e^{i\omega t} \left\{ \int_{-\infty}^{\infty} \mathcal{E}(t') e^{-i\omega t'} dt' \right\} d\omega \qquad (1.124)$$

Reversing the integration order and transforming the time variable as $\tau = t - t'$ yields

$$\mathcal{D}(t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} \left\{ \int_{-\infty}^{\infty} \varepsilon(\omega) e^{i\omega\tau} d\omega \right\} \mathcal{E}(t-\tau) d\tau \qquad (1.125)$$

Define

$$\mathcal{F}(\tau) = \frac{1}{2\pi}\int_{-\infty}^{\infty} \left[\varepsilon(\omega) - \varepsilon_0\right] e^{i\omega\tau} d\omega \qquad (1.126)$$

a Green's Function

Rewriting (1.125) as

$$\mathcal{D}(t)=\frac{1}{2\pi}\int_{-\infty}^{\infty}\left\{\int_{-\infty}^{\infty}\left[\varepsilon(\omega)-\varepsilon_0\right]e^{i\omega\tau}d\omega\right\}\mathcal{E}(t-\tau)d\tau$$

$$+\frac{1}{2\pi}\int_{-\infty}^{\infty}\left\{\int_{-\infty}^{\infty}\varepsilon_0 e^{i\omega\tau}d\omega\right\}\mathcal{E}(t-\tau)d\tau \qquad (1.127)$$

(1.126) into (1.127) yields

$$\mathcal{D}(t)=\int_{-\infty}^{\infty}\mathcal{F}(\tau)\mathcal{E}(t-\tau)d\tau+\frac{1}{2\pi}\int_{-\infty}^{\infty}\varepsilon_0 e^{i\omega t}\left\{\int_{-\infty}^{\infty}\mathcal{E}(t')e^{-i\omega t'}dt'\right\}d\omega$$

$$=\varepsilon_0\mathcal{E}(t)+\int_{-\infty}^{\infty}\mathcal{F}(\tau)\mathcal{E}(t-\tau)d\tau \qquad (1.128)$$

Now, (1.128) implies that $\mathcal{D}(t)$ depends on $\mathcal{E}$ at times other than t

→ a nonlocal relationship

We now invoke causality → there must be a beginning

$$\mathcal{F}(\tau)=0 \text{ for } \tau<0 \qquad (1.129)$$

Hence, (1.129) constrains (1.128) as

$$\mathcal{D}(t)=\varepsilon_0\mathcal{E}(t)+\int_{0}^{\infty}\mathcal{F}(\tau)\mathcal{E}(t-\tau)d\tau \qquad (1.130)$$

NB. For (1.130) to converge, $\mathcal{F}(\tau)$ must be a real function.

(1.121) & (1.122) into (1.130) yield

$$\int_{-\infty}^{\infty}\varepsilon(\omega)\mathbf{E}(\omega)e^{i\omega t}d\omega=\varepsilon_0\int_{-\infty}^{\infty}\mathbf{E}(\omega)e^{i\omega t}d\omega$$

$$+\int_{0}^{\infty}\mathcal{F}(\tau)\left\{\int_{-\infty}^{\infty}\mathcal{E}(\omega)e^{i\omega(t-\tau)}d\omega\right\}d\tau \qquad (1.131)$$

The last term in (1.131) may be written as

$$\int_{-\infty}^{\infty}\left\{\int_{0}^{\infty}\mathcal{F}(\tau)e^{-i\omega\tau}d\tau\right\}\boldsymbol{\mathcal{E}}(\omega)e^{i\omega t}d\omega \qquad (1.132)$$

(1.132) into (1.131) establishes the frequency dependence of $\varepsilon(\omega)$ through the integrand as

$$\varepsilon(\omega)=\varepsilon_0+\int_0^{\infty}\mathcal{F}(\tau)e^{-i\omega\tau}d\tau \qquad (1.133)$$

provided $\mathcal{F}(\tau)$ is known.

Since $\mathcal{F}(\tau)$ is a real function

$$\varepsilon^*(\omega)=\varepsilon(-\omega) \qquad (1.134)$$

Then, (1.117) into (1.134) implies

Even: $$\varepsilon'(-\omega)=\varepsilon'(\omega) \qquad (1.135)$$

Odd: $$\varepsilon''(-\omega)=-\varepsilon''(\omega) \qquad (1.136)$$

Transforming to the complex plane, $\omega \rightarrow z$, where $z = \omega + i\gamma$ for $\gamma > 0$ but small, (1.133) becomes

$$\varepsilon(z)=\varepsilon_0+\int_0^{\infty}\mathcal{F}(\tau)e^{-iz\tau}d\tau \qquad (1.137)$$

What does (1.137) reveal?

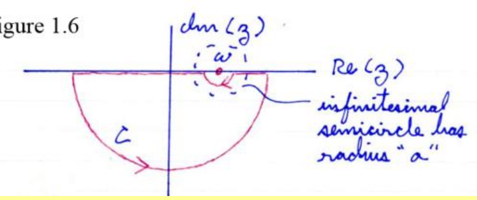
That $\varepsilon(z) - \varepsilon_0$ is an analytic function of z in the lower half plane.

i.e. It converges (diverges) in the lower (upper) half-plane

We can now consider a contour integral in the lower half plane about the pole $z = \omega$. From Cauchy's theorem

$$\oint_C \frac{\varepsilon(z)-\varepsilon_0}{z-\omega}dz=0 \qquad (1.138)$$

Figure 1.6



The semicircle portion of C does not contribute to (1.138) as $z \to \infty$. Hence, (1.138) becomes

$$0 = \lim_{a \to 0} \left\{ \int_{\infty}^{\omega+a} \frac{\varepsilon(z) - \varepsilon_0}{z - \omega} dz + \int_{\omega-a}^{-\infty} \frac{\varepsilon(z) - \varepsilon_0}{z - \omega} dz \right\} - i\pi \left[\varepsilon(\omega) - \varepsilon_0 \right] \quad (1.139)$$

Rewriting (1.139) as a principle value integral:

$$\varepsilon(\omega) - \varepsilon_0 = \frac{i}{\pi} \mathrm{P} \int_{-\infty}^{\infty} \frac{\varepsilon(\omega') - \varepsilon_0}{\omega' - \omega} d\omega' , \quad (1.140)$$

where $z \to \omega'$ since the integration is over the real axis.

Now, using (1.117) and equating real and imaginary components in (1.140) yields

$$\varepsilon'(\omega) - \varepsilon_0 = \frac{1}{\pi} \mathrm{P} \int_{-\infty}^{\infty} \frac{\varepsilon''(\omega')}{\omega' - \omega} d\omega' \quad (1.141)$$

$$\varepsilon''(\omega) = \frac{1}{\pi} \mathrm{P} \int_{-\infty}^{\infty} \frac{\varepsilon'(\omega) - \varepsilon_0}{\omega' - \omega} d\omega' \quad (1.142)$$

Since

$$\mathrm{P} \int_{-\infty}^{\infty} \frac{\varepsilon_0}{\omega' - \omega} d\omega' = 0 , \quad (1.143)$$

then (1.142) becomes

$$\varepsilon''(\omega) = \frac{1}{\pi} \mathrm{P} \int_{-\infty}^{\infty} \frac{\varepsilon'(\omega')}{\omega' - \omega} d\omega' \quad (1.144)$$

If we remove the singularities in (1.141) and (1.144), we get the Kramers-Kronig relations in the form

$$\varepsilon'(\omega) - \varepsilon_0 = \frac{1}{\pi} \int_{-\infty}^{\infty} \frac{\varepsilon''(\omega') - \varepsilon''(\omega)}{\omega' - \omega} d\omega' \quad (1.145)$$

$$\varepsilon''(\omega) = \frac{1}{\pi} \int_{-\infty}^{\infty} \frac{\varepsilon'(\omega') - \varepsilon'(\omega)}{\omega' - \omega} d\omega' \quad (1.146)$$

Using the symmetry properties of (1.135) and (1.136) gives the final forms for the Kramers-Kronig relations in terms of positive frequencies, as

$$\varepsilon'(\omega) - \varepsilon_0 = \frac{2}{\pi} \int_{0}^{\infty} \frac{\omega'\varepsilon''(\omega') - \omega\varepsilon''(\omega)}{(\omega')^2 - \omega^2} d\omega' \quad (1.147)$$

$$\varepsilon''(\omega) = \frac{2\omega}{\pi} \int_{0}^{\infty} \frac{\varepsilon'(\omega') - \varepsilon'(\omega)}{(\omega')^2 - \omega^2} d\omega' \quad (1.148)$$