MAT1330 DGD 4

1. Find the average rate of change of $f(t) = e^{5x}$ over the interval $[0, 1]$.

   Solution:
   $$\frac{\Delta y}{\Delta x} = \frac{y_2 - y_1}{x_2 - x_1} = \frac{f(1) - f(0)}{1 - 0} = e^5 - 1.$$

2. Calculate limits:

   $\lim_{x\to 0} \frac{x}{\sin 2x} = \lim_{x\to 0} \frac{1}{2}\frac{2x}{\sin 2x} = \frac{1}{2}$.

3. Let $y = \frac{2x^2-1}{x^2-9}$.

   (1) Find all the possible vertical asymptotes.

   (2) Find all the possible horizontal asymptotes.

   Solution: (i) From $x^2 - 9 = 0$ we have $x = 3$ and $x = -3$. VA: $x = 3$, $x = -3$.

   (ii)
   $$\lim_{x\to\infty} \frac{2x^2 - 1}{x^2 - 9} = 2$$
   So HA: $y = 2$.

4. Calculate the following limit $\lim_{x\to-\infty}(\sqrt{x^2 + x + 1} + x)$.

   Solution:
   $$\lim_{x\to-\infty}(\sqrt{x^2 + x + 1} + x) = \lim_{x\to-\infty} \frac{(\sqrt{x^2 + x + 1} + x)(\sqrt{x^2 + x + 1} - x)}{\sqrt{x^2 + x + 1} - x}$$
   $$= \lim_{x\to-\infty} \frac{x + 1}{\sqrt{x^2 + x + 1} - x} = \lim_{x\to-\infty} \frac{x + 1}{\sqrt{x^2(1 + \frac{1}{x} + \frac{1}{x^2})} - x}$$
   $$= \lim_{x\to-\infty} \frac{x(1 + \frac{1}{x})}{-x\sqrt{1 + \frac{1}{x} + \frac{1}{x^2}} - x}$$
   $$= \lim_{x\to-\infty} \frac{1 + \frac{1}{x}}{-\sqrt{1 + \frac{1}{x} + \frac{1}{x^2}} - 1} = -\frac{1}{2}.$$

5. Let

$$f(x) = \begin{cases} cx^2 - 2x & \text{if } x < 2 \\ cx - 8\cos[(x-1)\pi] & \text{if } x > 2 \\ 3k + 5 & \text{if } x = 2. \end{cases}$$

For what values of the constants c and k is the function $f(x)$ continuous at $x = 2$?

Solution: Note that $f(x)$ is continuous at $x = 2$ if

$$\lim_{x\to 2} f(x) = f(2).$$

Since

$$\lim_{x\to 2^-} f(x) = 4c - 4, \quad \lim_{x\to 2^+} f(x) = 8 + 2c.$$

When the limit $\lim_{x\to 2} f(x)$ exists, $4c - 4 = 8 + 2c, \Rightarrow c = 6$.

$$\lim_{x\to 2} f(x) = 4(6) - 4 = 20, \Rightarrow$$

$$3k + 5 = 20, \Rightarrow k = 5.$$

6. Identify the following limits as the derivative $f'(a)$.

(a) $\lim_{h\to 0} \frac{e^{3+h}-e^3}{h}$.

Solution: $f(x) = e^x$, $a = 3$.

(b) $\lim_{h\to 0} \frac{\sin(2\pi+h)}{h}$.

Solution: $f(x) = \sin x$, $a = 2\pi$.

(c) $\lim_{t\to 0} \frac{\sqrt{9+t}-3}{t}$.

Solution: $f(x) = \sqrt{x}$, $a = 9$.

7. Let $f(x) = x^3$. Calculate $f'(5)$ by DEFINITION.

Solution:

$$f'(5) = \lim_{h\to 0} \frac{f(5+h) - f(5)}{h} = \lim_{h\to 0} \frac{(5+h)^3 - 5^3}{h} = \lim_{h\to 0} \frac{5^3 + 3(5^2)h + 3(5)h^2 + h^3 - 5^3}{h}$$

$$= \lim_{h\to 0} \frac{3(5^2)h + 3(5)h^2 + h^3}{h} = \lim_{h\to 0} (3(5^2) + 3(5)h + h^2) = 75.$$

8. Let $f(x) = -\dfrac{1}{x^2}$. Calculate $f'(3)$ by the definition.

Solution:

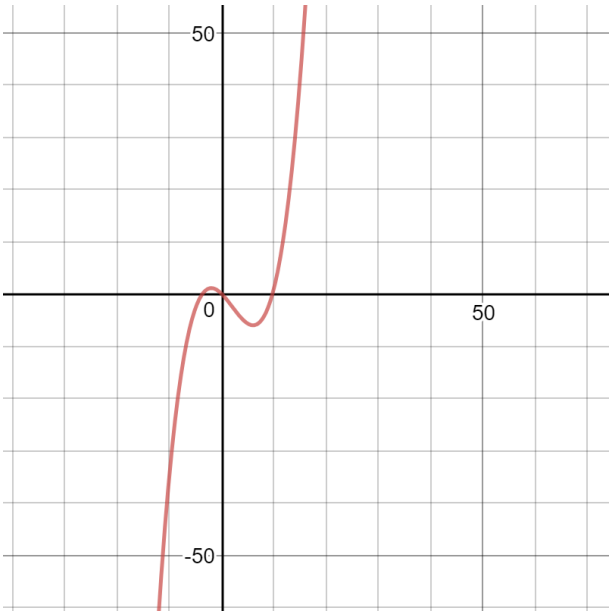
$$f'(3) = \lim_{h\to 0} \frac{f(3+h) - f(3)}{h} = \lim_{h\to 0} \frac{\frac{1}{9} - \frac{1}{(h+3)^2}}{h} = \lim_{h\to 0} \frac{\frac{(h+3)^2-9}{9(h+3)^2}}{h} = \lim_{h\to 0} \frac{h^2+6h}{9(h+3)^2 h}$$

$$= \lim_{h\to 0} \frac{h+6}{9(h+3)^2} = \frac{6}{81} = \frac{2}{27}.$$

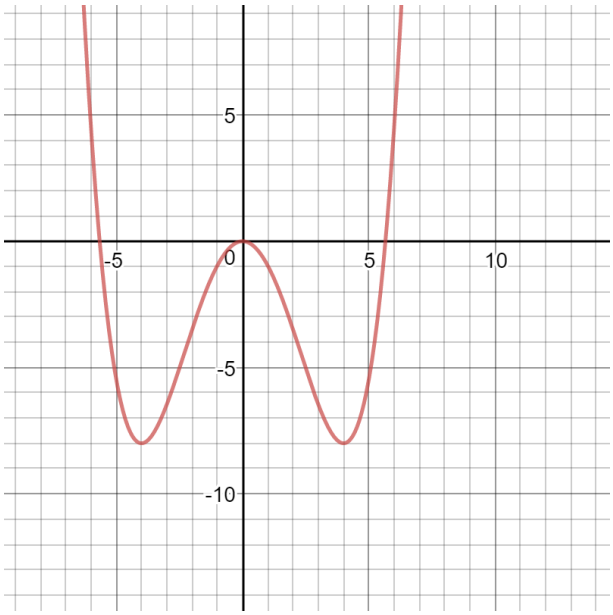
9. Sketch the graph of the following continuous functions by finding critical numbers and intervals of increase and decrease, horizontal tangent lines:

(a) $f'(x) > 0$ when $x < -2$ and $x > 6$; $f'(x) < 0$ when $-2 < x < 6$; $f'(-2) = f(6) = 0$.

Solution: Critical numbers are -2, 6. Increase when $x < -2$ and $x > 6$, and decrease when $-2 < x < 6$. Horizontal tangent lines are $y = -2$ and $y = 6$.



(b) $f'(x) > 0$ when $-4 < x < 0$ and $x > 4$; $f'(x) < 0$ when $x < -4$ and $0 < x < 4$; $f'(-4) = f(4) = 0$.



Solution: Critical numbers are -4, 4. Increase when $-4 < x < 0$ and $x > 4$; decrease when $x < -4$ and $0 < x < 4$. Horizontal tangent lines are $y = -4$ and $y = 4$.