

11. Power Series

- ◇ We have now see that some infinite series $\sum a_n$ converge.
- ◇ Now we consider *power series* which are essentially polynomials with infinitely many terms.

POWER SERIES

A **power series centered at** $x = a$ has the form

$$\sum_{n=0}^{\infty} C_n(x-a)^n = C_0 + C_1(x-a) + C_2(x-a)^2 + C_3(x-a)^3 + \dots$$

Notice that when we plug in a particular value, say $x = p$, we get

$$\sum_{n=0}^{\infty} C_n(p-a)^n = C_0 + C_1(p-a) + C_2(p-a)^2 + C_3(p-a)^3 + \dots$$

which is just a numerical series $\sum a_n$ like what we've already been studying, but with $a_n = C_n(p-a)^n$ for $n=0, 1, 2, \dots$

Thus, a power series may be a convergent series for some values of x .

In particular, if $x = a$ (the **centre**), then we have

$$\sum_{n=0}^{\infty} C_n(a-a)^n = C_0 + C_1(0) + C_2(0)^2 + \dots = C_0$$

⇒ All power series converge at their centre $x=a$.

- In general, the convergence of a power series will depend on the **coefficients**, the c_j 's
- If a power series is centred at $a = 0$, then it looks like this:

$$\sum_{n=0}^{\infty} C_n(x-0)^n = \sum_{n=0}^{\infty} C_n x^n = C_0 + C_1 x + C_2 x^2 + C_3 x^3 + \dots$$

- A power series defines a function! What is its domain?

$$f(x) = \sum_{n=0}^{\infty} C_n(x-a)^n$$

↳ all x for which the series $f(x)$ is convergent.

Example 11.1. Suppose a power series has coefficients $c_j = 1$ for all j and is centred at $a = 0$. What does the series look like and what is its domain?

If $c_j = 1$ for all j and the centre is $a = 0$, then the series we get is

$$f(x) = \sum_{n=0}^{\infty} x^n = 1 + x + x^2 + x^3 + x^4 + \dots$$

for each $x \in \mathbb{R}$, this will be a geometric series !

Ex $f(\frac{1}{2}) = 1 + \frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \dots = \sum_{n=0}^{\infty} (\frac{1}{2})^n$ which converges to $\frac{1}{1-\frac{1}{2}} = 2$

but $f(2)$ is undefined since $\sum_{n=0}^{\infty} 2^n$ diverges.

In general, $\sum_{n=0}^{\infty} x^n \begin{cases} \text{converges to } \frac{1}{1-x} & \text{if } |x| < 1 \\ \text{diverges} & \text{if } |x| \geq 1. \end{cases}$

\therefore the domain of $f(x) = \sum_{n=0}^{\infty} x^n$ is all x such that $|x| < 1$

Remark. We can use the Ratio Test to find values of x for which a power series is convergent!

Example 11.2. For what values of x is the power series $\sum_{n=1}^{\infty} \frac{x^n}{n^2 2^n}$ convergent? $a_n = \frac{x^n}{n^2 2^n}$

The Ratio Test can tell us most of the answer:

1. Find $\lim_{n \rightarrow \infty} \left| \frac{a_{n+1}}{a_n} \right|$
2. Solve for x when limit < 1

$$\begin{aligned} 1. \lim_{n \rightarrow \infty} \left| \frac{a_{n+1}}{a_n} \right| &= \lim_{n \rightarrow \infty} \left| \frac{x^{n+1}}{(n+1)^2 2^{n+1}} \bigg/ \frac{x^n}{n^2 2^n} \right| \\ &= \lim_{n \rightarrow \infty} \left| \frac{x^{n+1} \cdot n^2 \cdot 2^n}{x^n (n+1)^2 2^{n+1}} \right| \\ &= \lim_{n \rightarrow \infty} \left| \frac{x \cdot n^2}{2(n+1)^2} \right| \end{aligned}$$

$$\begin{aligned} &= \lim_{n \rightarrow \infty} \frac{1}{2} \cdot \frac{n^2}{(n+1)^2} |x| \\ &= \frac{1}{2} |x| \end{aligned}$$

Knowing $\lim_{n \rightarrow \infty} \left| \frac{a_{n+1}}{a_n} \right| = \frac{1}{2}|x|$, the Ratio Test tells us that

• the power series converges when $\frac{1}{2}|x| < 1 \Leftrightarrow |x| < 2$
or $-2 < x < 2$

• the power series diverges when $\frac{1}{2}|x| > 1$

What if $\lim_{n \rightarrow \infty} \left| \frac{a_{n+1}}{a_n} \right| = 1$? i.e. $\frac{1}{2}|x| = 1 \Leftrightarrow |x| = 2 \Leftrightarrow x = \pm 2$

↳ Ratio Test is inconclusive, so we check these cases separately:

• if $x = -2$ then $\sum_{n=0}^{\infty} \frac{x^n}{n^2 2^n} = \sum_{n=0}^{\infty} \frac{(-2)^n}{n^2 2^n} = \sum_{n=0}^{\infty} \frac{(-1)^n 2^n}{n^2 2^n} = \sum_{n=0}^{\infty} \frac{(-1)^n}{n^2}$ which converges (by A.S.T.)

• if $x = 2$ then $\sum_{n=0}^{\infty} \frac{x^n}{n^2 2^n} = \sum_{n=0}^{\infty} \frac{2^n}{n^2 2^n} = \sum_{n=0}^{\infty} \frac{1}{n^2}$ which converges (p-series $p=2$)

Conclusion: the power series $\sum_{n=0}^{\infty} \frac{x^n}{n^2 2^n}$ converges for $-2 \leq x \leq 2$.

RADIUS & INTERVAL OF CONVERGENCE

Theorem 11.3. For a power series $\sum_{n=0}^{\infty} c_n(x-a)^n$ there are only 3 possibilities:

1. The series converges only when $x=a$.
2. The series converges for all $x \in \mathbb{R}$.
3. There is a positive number R such that
 - the series converges if $|x-a| < R$ and
 - the series diverges if $|x-a| > R$.

R is called the radius of convergence of the power series.

In Ex. 11.2 we had $R=2$.

Conventions for R and the Interval of Convergence

1. If we're in Case 1, we write $R=0$
2. If we're in Case 2, we write $R=\infty$
3. If we're in Case 3, then

R defines the interval of convergence $|x-a| < R$

or $a-R < x < a+R$ on which the series converges:

But we must check each endpoint $a-R$ and $a+R$ separately.

In Ex. 11.2, the interval of convergence was $-2 \leq x \leq 2$

- In Case 1, we write $\{a\}$ for the interval of convergence
 - In Case 2, we write $(-\infty, \infty)$ for the interval of convergence
-

Example 11.4. Find the radius and interval of convergence of each of the following power series:

i. $\sum_{n=1}^{\infty} \frac{\ln n}{n} (x+1)^n \leftarrow a_n = \frac{\ln(n)}{n} (x+1)^n$ and centre is $a = -1$

Ratio Test: $\lim_{n \rightarrow \infty} \left| \frac{a_{n+1}}{a_n} \right| = \lim_{n \rightarrow \infty} \left| \frac{\frac{\ln(n+1)}{n+1} (x+1)^{n+1}}{\frac{\ln(n)}{n} (x+1)^n} \right|$

$$= \lim_{n \rightarrow \infty} \frac{n \ln(n+1)}{(n+1) \ln(n)} |x+1|$$

$$= |x+1| \quad \left(\text{since } \lim_{n \rightarrow \infty} \frac{n}{n+1} = 1 \text{ and } \lim_{n \rightarrow \infty} \frac{\ln(n+1)}{\ln(n)} = 1 \right) \leftarrow \text{check!}$$

\Rightarrow series will converge for $|x+1| < 1 \iff -1 < x+1 < 1$

$\therefore R = 1$ \leftarrow or $-2 < x < 0$

check endpoints of interval:

• if $x = -2$, $\sum_{n=1}^{\infty} \frac{\ln n}{n} (x+1)^n = \sum_{n=1}^{\infty} \frac{\ln n}{n} (-2+1)^n = \sum_{n=1}^{\infty} \frac{(-1)^n \ln n}{n} \leftarrow \text{converges (A.S.T.)}$
 \leftarrow check!

• if $x = 0$, $\sum_{n=1}^{\infty} \frac{\ln n}{n} (x+1)^n = \sum_{n=1}^{\infty} \frac{\ln n}{n} (0+1)^n = \sum_{n=1}^{\infty} \frac{\ln n}{n} \leftarrow \text{diverges } (> \sum \frac{1}{n})$
 \leftarrow check!

\therefore the interval of convergence is $[-2, 0)$ or $-2 \leq x < 0$

ii. $\sum_{n=1}^{\infty} \frac{n!x^n}{n^3} \leftarrow a_n = \frac{n!x^n}{n^3}$ and centre is $a=0$

$$\begin{aligned} \text{Ratio Test: } \lim_{n \rightarrow \infty} \left| \frac{a_{n+1}}{a_n} \right| &= \lim_{n \rightarrow \infty} \left| \frac{(n+1)!x^{n+1}}{(n+1)^3} \bigg/ \frac{n!x^n}{n^3} \right| \\ &= \lim_{n \rightarrow \infty} \frac{(n+1)n^3}{(n+1)^3} |x| \\ &= \infty \end{aligned}$$

∴ this series diverges for all x except its centre $x=0$.

∴ $R=0$ and interval of convergence is $\{a\} = \{0\}$

iii. $\sum_{n=0}^{\infty} \frac{(-1)^n x^{2n}}{(2n)!} \leftarrow a_n = \frac{(-1)^n x^{2n}}{(2n)!}$ and centre is $x=0$.

$$\begin{aligned} \text{Ratio Test: } \lim_{n \rightarrow \infty} \left| \frac{a_{n+1}}{a_n} \right| &= \lim_{n \rightarrow \infty} \left| \frac{(-1)^{n+1} x^{2(n+1)}}{(2(n+1))!} \bigg/ \frac{(-1)^n x^{2n}}{(2n)!} \right| \\ &= \lim_{n \rightarrow \infty} \left| \frac{(-1)^{n+1} x^{2n+2} (2n)!}{(-1)^n x^{2n} (2n+2)!} \right| \\ &= \lim_{n \rightarrow \infty} \left| \frac{-x^2 \cancel{(2n)!}}{(2n+2)(2n+1)\cancel{(2n)!}} \right| \\ &= \lim_{n \rightarrow \infty} \frac{x^2}{(2n+2)(2n+1)} \\ &= 0 \end{aligned}$$

∴ this series is convergent for all x .

∴ $R = \infty$ and $(-\infty, \infty)$

radius
of
convergence

interval
of
convergence

STUDY GUIDE

Power Series

Radius of Convergence R

Interval of Convergence