Column space, row space and null space.

(12-1)

A $m \times n$ matrix

<u>Def.</u> The column space of A (also called the image of A, im(A)) is

$$\underset{\text{Col}(A)}{\overset{\|}{\text{im}(A)}} = \text{Span}\{\underbrace{c_1, c_2, \ldots, c_n}_{\text{column vectors}}\}$$

<u>Recall</u>: Ax, $x \in \mathbb{R}^n$ is a linear comb of columns

(12-2)

<u>Def</u>. The row space of A is

$$\text{Row}(A) = \text{Span}\{r_1, r_2, \ldots, r_m\}$$

where $\{r_1, r_2, \ldots, r_m\}$ are rows of A

<u>Def</u>. The null space of A (also the kernel of A) is

$$\underset{\ker(A)}{\overset{\|}{\text{Null}(A)}} = \{x \in \mathbb{R}^n \mid Ax = 0\}$$

12-3

<u>Theorem</u> Null(A) is a subspace in $\mathbb{R}^n$

<u>Proof</u> 1. $0 \in Null(A)$

2. $v, u \in Null(A) \iff Av = Au = 0$.

so $A(v+u) = 0$ $\quad v+u \in Null(A)$

3. $v \in Null(A)$ $\quad c \in \mathbb{R}$ $\quad Av = 0$

$A(cv) = c(Av) = c \cdot 0 = 0$ $\quad cv \in Null(A)$

Basis for Null(A) 12-4

<u>Ex.</u>

$$\left[\begin{array}{cccc|c} 1 & 2 & 3 & -3 & 0 \\ 0 & 1 & 1 & -2 & 0 \\ 1 & 0 & 1 & 1 & 0 \end{array}\right] \rightsquigarrow \left[\begin{array}{cccc|c} 1 & 0 & 1 & 1 & 0 \\ 0 & 1 & 1 & -2 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right]$$

RREF

Null(A) does not change under elementary transformation (operations)

Null(A) = the set of solutions of the homogeneous system

(12-5)

$$Null(A) = \left\{ \begin{bmatrix} -r-t \\ -r+2t \\ r \\ t \end{bmatrix} \;\middle|\; r, t \in \mathbb{R} \right\} =$$

$$= span \left\{ \begin{bmatrix} -1 \\ -1 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} -1 \\ 2 \\ 0 \\ 1 \end{bmatrix} \right\}.$$

$\uparrow$

basis as it is L.I.

(12-6)

<u>Basic</u> solutions are the basis vectors of $Null(A)$.

<u>Theorem</u>. (Rank-Nullity).

~~The~~ $\dim(Null(A)) + rank(A) = n.$

<u>Proof</u> ...

Inhomogeneous Systems (12-7)

$$A = \begin{bmatrix} 1 & 2 & 3 & -3 \\ 0 & 1 & 1 & -2 \\ 1 & 0 & 1 & 1 \end{bmatrix} \qquad b = \begin{bmatrix} 10 \\ 3 \\ 4 \end{bmatrix}.$$

$$[A|b] \rightsquigarrow \left[\begin{array}{cccc|c} 1 & 0 & 1 & 1 & 4 \\ 0 & 1 & 1 & -2 & 3 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right]. \quad \text{RREF}$$

Gen Solution is $\left\{ \begin{bmatrix} 4-r-t \\ 3-r+2t \\ r \\ t \end{bmatrix} \;\middle|\; r,t \in \mathbb{R} \right\}$

(12-8.)

That is

$$\left\{ \underset{\text{extra solution.}}{\begin{bmatrix} 4 \\ 3 \\ 0 \\ 0 \end{bmatrix}} + \underbrace{r \begin{bmatrix} -1 \\ -1 \\ 1 \\ 0 \end{bmatrix} + t \begin{bmatrix} -1 \\ 2 \\ 0 \\ 1 \end{bmatrix}}_{\text{Null space.}} \;\middle|\; r,t \in \mathbb{R} \right\}$$

= One solution + all solutions from Null(A).