MATH 3705* A Test 1 Solutions January 2019

Questions 1-5 are multiple choice. Circle the correct answer. Only the answer will be marked. [**12 marks**].

1. [**2**] $\mathcal{L}\{e^{-4t}\cos(3t)\} =$

(a) $\dfrac{s+4}{(s+4)^2+9}$ (b) $\dfrac{4s}{s^2+9}$ (c) $\dfrac{s-4}{(s-4)^2+9}$ (d) None of the above

2. [**2**] $\mathcal{L}\{t\sin(3t)\} =$

(a) $\dfrac{6s}{(s^2+9)^2}$ (b) $\dfrac{s^2-9}{(s^2+9)^2}$ (c) $\dfrac{9-s^2}{(s^2+9)^2}$ (d) None of the above

3. [**2**] $\mathcal{L}\{u(t-1)\,e^{2t}\} =$

(a) $\dfrac{e^{-s}}{s-2}$ (b) $\dfrac{e^{1-s}}{s+2}$ (c) $\dfrac{e^{2-s}}{s-2}$ (d) None of the above

4. [**3**] $\mathcal{L}^{-1}\left\{\dfrac{2s}{s^2-25}\right\} =$

(a) $e^{5t}-e^{-5t}$ (b) $e^{5t}+e^{-5t}$ (c) $-\cos(5t)$ (d) None of the above

5. [**3**] $\mathcal{L}^{-1}\left\{\dfrac{3s+6}{s^2-4s+13}\right\} =$

(a) $3e^{2t}\cos(3t)+4e^{2t}\sin(3t)$ (b) $3e^{-2t}\cos(3t)+\dfrac{9}{4}e^{-2t}\sin(3t)$

(c) $3e^{2t}\cos(3t)+\dfrac{9}{4}e^{2t}\sin(3t)$ (d) None of the above

Answers: 1.(a), 2.(a), 3.(c), 4.(b), 5.(a).

6. [**4 marks**] Let $f(t) = e^t$ for $0 < t < 1$ and $f(t+1) = f(t)$ for all $t \geq 0$. Find $\mathcal{L}\{f(t)\}$.

Solution:

Since f is periodic with the period $\omega = 1$, then

$$\mathcal{L}\{f(t)\} = \frac{1}{1-e^{-s}} \int_0^1 e^t\, e^{-st}\, dt = \frac{1}{1-e^{-s}} \int_0^1 e^{t(1-s)}\, dt = \frac{1}{1-e^{-s}} \left\{ \frac{1}{1-s}\, e^{t(1-s)} \right\}_0^1 =$$

$$= \frac{1}{1-e^{-s}} \left(\frac{1}{1-s} \right) (e^{1-s} - e^0) = \frac{e^{1-s} - 1}{(1-e^{-s})(1-s)}.$$

7. [**7 marks**] Employ the Laplace transform to solve the initial-value problem

$$y'' - 6y' + 13y = 0, \quad y(0) = 0,\ y'(0) = 5.$$

Solution:

Take the Laplace transform of both parts of the equation:

$$[s^2 Y(s) - sy(0) - y'(0)] - 6[sY(s) - y(0)] + 13Y(s) = 0.$$

Substitute $y(0) = 0,\ y'(0) = 5$ and solve for $Y(s)$:

$$(s^2 - 6s + 13)Y(s) = 5, \Rightarrow Y(s) = \frac{5}{s^2 - 6s + 13} = \frac{5}{(s-3)^2 + 4}.$$

Using the First Shifting Theorem with $a = 3$, find

$$\mathcal{L}^{-1}\{Y(s)\} = 5\mathcal{L}^{-1}\left\{ \frac{1 \cdot 2 \cdot \frac{1}{2}}{(s-3)^2 + 4} \right\} = \frac{5}{2} e^{3t} \sin(2t) = y(t).$$

8. [**7 marks**] Find $f(t)$, if

$$f(t) - 2\int_0^t f(x)\,dx = 8t.$$

Solution:

Let $\mathcal{L}\{f(t)\} = F(s)$. Take the Laplace transform of both parts of the equation:

$$F(s) - 2\frac{1}{s}F(s) = 8\frac{1}{s^2} \quad \Rightarrow F(s)\left(1 - \frac{2}{s}\right) = \frac{8}{s^2} \quad \Rightarrow F(s) = \frac{8}{s(s-2).}$$

To invert $F(s)$, one can use **(a)** partial fractions or **(b)** $\mathcal{L}\left\{\int_0^t g(x)\,dx\right\} = \frac{G(s)}{s}.$

(a) $F(s) = \dfrac{8}{s(s-2)} = -\dfrac{4}{s} + \dfrac{4}{s-2} \quad \Rightarrow \quad y(t) = -4 + 4e^{2t}.$

OR

(b) $\dfrac{8}{s(s-2)} = \dfrac{1}{s} \cdot \dfrac{8}{s-2} = \dfrac{1}{s} \cdot G(s)$. Then $\mathcal{L}^{-1}\left\{G(s)\right\} = \mathcal{L}^{-1}\left\{\dfrac{8}{s-2}\right\} = 8e^{2t} = g(t),$

and

$$y(t) = \int_0^t g(x)\,dx = 8\int_0^t e^{2x}\,dx = 8 \cdot \frac{1}{2}e^{2x}\Big|_0^t = 4e^{2t} - 4.$$