

MAT1332 C: Instructor Wan-Yu Tsai

Wednesday, February 13, 2019 : Test #1

Duration: 75 minutes

Family name: _____

First name: _____

Student number : _____

For picking up your graded test:

Circle the DGD you will attend to pick up your test (whether you are registered or not). Tests take at least one week to grade; watch for announcements in class.


# :	DGD1	DGD2	DGD3	DGD4
Day :	Thursday	Thursday	Thursday	Friday
Start time:	11:30	11:30	17:30	11:30
Room :	VNR 1075	MRT 221	STE C0136	SMD 221


Please read the following instructions carefully.

- You have 75 minutes to complete this exam. It is worth 20% of your final grade.
- This is a closed book exam. Except for Faculty-approved calculators (models: Texas Instruments TI-30* and TI-34*, Casio FX-260* and Casio FX-300*), no notes, cell phones, smartwatches or related devices of any kind are permitted. All such devices, including cell phones, **must be stored in your bag under your desk for the duration of the exam.**
- Read each question carefully — you will save yourself time and grief later on.
- Questions 1 through 4 are a combination of multiple choice and short answer, worth a total of 9 points. For multiple choice, **circle your answer**; otherwise, give your answers in the spaces provided.
- Questions 5 through 7 are long answer, with number of points as indicated. **You must show your work and your work must be legible and well-justified, to earn full credit.**
- Where it is possible to check your work, do so.
- Good luck!


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Marker's use only:

Question	Marks
1,2 (/4)	
3,4 (/5)	
5 (/5)	
6 (/4)	
7 (/9)	
Total (/27)	

1. (2 points) Suppose we wish to estimate the value of $\int_1^3 \sin(x^2) dx$. The corresponding left Riemann sum on 4 intervals is (circle one):

A. $\sum_{i=0}^3 \sin(x_i^2) \Delta x$

C. $\sum_{i=1}^4 \cos(x_i^2) \Delta x$

E. $\sum_{i=1}^3 x_i \sin(\Delta x^2)$

B. $\sum_{i=1}^4 \sin(x_i^2) \Delta x$

D. $\sum_{i=0}^3 \cos(x_i^2) \Delta x$

F. $\sum_{i=0}^4 x_i \cos(\Delta x^2)$

where the values of Δx , x_0 and x_1 are, respectively (circle one):

A. $\Delta x = 2, x_0 = 1, x_1 = 3$

D. $\Delta x = 3, x_0 = 0, x_1 = 1.5$

B. $\Delta x = 0.5, x_0 = 1, x_1 = 1.5$

E. $\Delta x = 0.5, x_0 = 0, x_1 = 0.5$

C. $\Delta x = 1, x_0 = 1, x_1 = 2$

F. $\Delta x = 1, x_0 = 0, x_1 = 1$

Solution: $\Delta x = \frac{3-1}{4} = 0.5$ so answers are A and B

2. (2 points) First evaluate the indefinite integral $\int \frac{4x}{e^{x^2}} dx$.

(a) Answer:

Solution: Let $u = x^2$ and $du = 2x dx$ so

$$\int \frac{4x}{e^{x^2}} dx = \int \frac{4}{e^u} \cdot \frac{1}{2} du = -2e^{-u} + c = -2e^{-x^2} + c$$

for an arbitrary constant c .

Then evaluate, if possible, the improper integral $\int_0^\infty \frac{4x}{e^{x^2}} dx$.

(b) The integral (circle one):

A. converges to 2

D. diverges to $-\infty$

B. diverges to ∞

E. converges to 3

C. converges to 1.5

F. is impossible to evaluate because ∞ is not a number.

Solution:

$$\begin{aligned} \int_0^{\infty} \frac{4x}{e^{x^2}} dx &= \lim_{T \rightarrow \infty} \int_0^T \frac{4x}{e^{x^2}} dx \\ &= \lim_{T \rightarrow \infty} \left[-2e^{-x^2} \right]_0^T \\ &= \lim_{T \rightarrow \infty} \left(-2e^{-T^2} + \frac{3}{2} \right) \\ &= 2 \end{aligned}$$

so the integral converges, to 2

3. (2 points) The polar form of the complex number $u = -3 - 3i$ is $u = re^{i\varphi}$ where (circle one):

- | | | |
|--------------------|-------------|---------------------|
| A. $r = 3$ | C. $r = 9$ | E. $r = -3$ |
| B. $r = 3\sqrt{2}$ | D. $r = 18$ | F. $r = -3\sqrt{2}$ |

and (circle one):

- | | | |
|----------------------|------------------------|------------------------|
| A. $\varphi = i$ | C. $\varphi = 3\pi/4$ | E. $\varphi = -2\pi/3$ |
| B. $\varphi = \pi/4$ | D. $\varphi = -3\pi/4$ | F. $\varphi = -\pi$. |

Solution: $|u| = \sqrt{(-3)^2 + (-3)^2} = \sqrt{18} = 3\sqrt{2} = r$. Thus

$$-3 = r \cos(\varphi) \quad \text{and} \quad -3 = r \sin(\varphi)$$

imply that $\cos(\varphi) = \frac{-3}{3\sqrt{2}} = \frac{-1}{\sqrt{2}}$ and $\sin(\varphi) = \frac{-3}{3\sqrt{2}} = \frac{-1}{\sqrt{2}}$. Thus $\varphi = -3\pi/4$.
The answers are B and D.

4. (3 points) If $A = \begin{bmatrix} 1 & 2 \\ -1 & -3 \end{bmatrix}$, $B = \begin{bmatrix} -1 & 0 & 1 \\ 0 & 1 & 2 \end{bmatrix}$, and $C = \begin{bmatrix} 1 & 3 \\ 4 & 5 \end{bmatrix}$, then compute the following matrices or else explain why they don't exist:

(a) $B^T A =$

(b) $BC =$

(c) $A - 2C =$

Solution: (a) We compute

$$B^T A = \begin{bmatrix} -1 & 0 \\ 0 & 1 \\ 1 & 2 \end{bmatrix} \begin{bmatrix} 1 & 2 \\ -1 & -3 \end{bmatrix} = \begin{bmatrix} -1 & -2 \\ -1 & -3 \\ -1 & -4 \end{bmatrix}$$

(b) Since B is 2×3 and C is 2×2 , the number of columns of B is not equal to the number of rows of C , so the product BC is not defined.

(c) We compute

$$A - 2C = \begin{bmatrix} 1 & 2 \\ -1 & -3 \end{bmatrix} - 2 \begin{bmatrix} 1 & 3 \\ 4 & 5 \end{bmatrix} = \begin{bmatrix} 1 & 2 \\ -1 & -3 \end{bmatrix} + \begin{bmatrix} -2 & -6 \\ -8 & -10 \end{bmatrix} = \begin{bmatrix} -1 & -4 \\ -9 & -13 \end{bmatrix}.$$

5. (5 points) Use the method of partial fractions to find the indefinite integral

$$\int \frac{3x^2 + 7x + 4}{x^2 + 2x - 8} dx.$$

Show your work and draw a box around your final answer.

Solution: We perform long division and factor the denominator to obtain

$$\frac{3x^2 + 7x + 4}{x^2 + 2x - 8} = 3 + \frac{x + 28}{(x + 4)(x - 2)}$$

Then solving

$$\frac{x + 28}{(x + 4)(x - 2)} = \frac{A}{x + 4} + \frac{B}{x - 2}$$

yields $A = -4$ and $B = 5$. Therefore

$$\begin{aligned} \int \frac{3x^2 + 7x + 4}{x^2 + 2x - 8} dx &= \int \left(3 + \frac{-4}{x + 4} + \frac{5}{x - 2} \right) dx \\ &= 3x - 4 \ln |x + 4| + 5 \ln |x - 2| + c \end{aligned}$$

for an arbitrary constant c .

6. (4 points) Determine the solution to the differential equation

$$y' = 5 \sin(x)(2 + 3y)^2$$

which satisfies the initial condition $y(0) = \frac{1}{3}$. Show your work and draw a box around your final answer, which should be an equation of the form $y = f(x)$.

Solution: This is a separable differential equation. We solve it as

$$\begin{aligned} \frac{1}{(2+3y)^2} y' &= 5 \sin(x) \\ \int \frac{1}{(2+3y)^2} dy &= \int 5 \sin(x) dx \\ \int \frac{1}{u^2} \left(\frac{1}{3}\right) du &= 5(-\cos(x)) + c \quad \text{where } u = 2 + 3y, du = 3 dy \\ \frac{1}{3} (-u^{-1}) &= -5 \cos(x) + c \quad (c \text{ is an arbitrary constant}) \\ -\frac{1}{3}(2+3y)^{-1} &= -5 \cos(x) + c \\ (2+3y)^{-1} &= 15 \cos(x) - 3c \\ 2+3y &= \frac{1}{15 \cos(x) - 3c} \\ 3y &= -2 + \frac{1}{15 \cos(x) - 3c} \\ y &= -\frac{2}{3} + \frac{1}{45 \cos(x) + d} \end{aligned}$$

where d is an arbitrary constant. Now when $x = 0$ we need $y = \frac{1}{3}$; this gives the equation

$$\frac{1}{3} = -\frac{2}{3} + \frac{1}{45 + d}$$

so $\frac{1}{45+d} = 1$ and $45 + d = 1$ so $d = -44$. Thus our answer is

$$y = -\frac{2}{3} + \frac{1}{45 \cos(x) - 44}$$

7. (9 points) The number of fish $N(t)$ on a fish farm varies according to the differential equation

$$\frac{dN}{dt} = \frac{N^3 - 8N^2 + 15N}{3N^2 + 1},$$

for t measured in months and $N(t)$ represents the amount of fish, in hundreds of kilograms, at time t . Answer the questions below, showing all work and putting a box around your final answer.

(a) (2 points) Find all equilibria of this differential equation and decide if they are biologically relevant.

Solution: Here, $f(N) = \frac{N^3 - 8N^2 + 15N}{3N^2 + 1}$. We must find where $f(N) = 0$. This happens with the numerator is 0. We factor

$$N^3 - 8N^2 + 15N = N(N - 5)(N - 3)$$

so the three equilibria are 0, 3 and 5, each of which are ≥ 0 therefore biologically relevant.

(b) (2 points) Draw a phase line diagram (phase portrait) for this differential equation.

Solution: (Including $N < 0$ in the diagram is optional, but of course irrelevant to the case at hand.)

N	$(-\infty, 0)$	$(0, 3)$	$(3, 5)$	$(5, \infty)$
$f(N)$	-	+	-	+

So the phase line diagram is a picture equivalent to:



(c) (3 points) State the stability theorem (fully and precisely, in complete sentences), and use it to determine the stability of the equilibrium $N^* = 0$.

Solution: The Stability Theorem states that if $y' = f(y)$ is an autonomous differential equation and y^* is an equilibrium, then (a) y^* is stable if $f'(y^*) < 0$ and (b) y^* is unstable if $f'(y^*) > 0$.

Here, we have $f(N) = \frac{N^3 - 8N^2 + 15N}{3N^2 + 1}$ so

$$f'(N) = \frac{(3N^2 + 1)(3N^2 - 16N + 15) - (N^3 - 8N^2 + 15N)(6N)}{(3N^2 + 1)^2}$$

so at $N^* = 0$, we have $f'(0) = 15 > 0$ so 0 is unstable.

(d) (1 point) Is your answer in (c) consistent with your phase portrait in (b)? Explain.

Solution: Yes. An equilibrium is unstable if the arrows on either side point away from it on the phase line diagram; this is the case of 0 in (b) and we saw the point was unstable in (c).

(e) (1 point) If the fish farm starts at $t = 0$ with $N(0) = 4$ hundreds of kilograms of fish, how many fish will it have in the long term?

Solution: Using our phase line diagram, we deduce that if we begin at $N(0) = 4$, then the population will decrease and in the long term will approach the stable fixed point of 3 (hundreds of kilograms of fish).