

## Solution to Review Questions for Midterm 2

MAT1341, Summer 2017

This midterm test covers Chapters 4-10, 14 and 18

Also see Linear Independence Examples and True-False questions posted before.

1. Determine whether each of the following statements is true or false.

(1) If  $\dim(V) = 3$ , then a set of vectors  $\{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3, \mathbf{v}_4\}$  in  $V$  is linearly dependent.

(T) If the number of vectors in a set is more than the dimension of the vector space, this set is linearly dependent.

(2) If a set of vectors  $S = \{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3, \mathbf{v}_4\}$  in a vector space  $V$  with  $\dim V = 5$ , then  $S$  is linearly independent.

(F) If the number of vectors in a set is less than the dimension of the vector space, this set is not necessarily linearly independent. For instance, if one of these vectors is the zero vector, then this set is linearly dependent.

(3) If a set of vectors  $S = \{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3, \mathbf{v}_4\}$  in a vector space  $V$  with  $\dim V = 4$ , then  $S$  is linearly independent.

(F) Only if  $\text{span } S = V$ , then  $S$  is a basis of  $V$ , and it is linearly independent.

(4) If a set of vectors  $S = \{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3, \mathbf{v}_4\}$  in a vector space  $V$  with  $\dim V = 4$ , then  $\text{span}(S) = V$ .

(F) Only if  $S$  is linearly independent, then  $S$  is a basis of  $V$ , and  $\text{span } S = V$ .

(5) If a set of vectors  $S = \{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3, \mathbf{v}_4\}$  in a vector space  $V$  is linearly independent and  $\text{span}(S) = V$ , then  $\dim(V) = 4$ .

(T)  $S$  is a basis of  $V$  so that  $\dim(V) = |S| = 4$ .

(6) If a set of vectors  $S = \{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3, \mathbf{v}_4\}$  in a vector space  $V$  with  $\dim(V) = 4$  is linearly independent, then  $\text{span}(S) = V$ .

(T)  $S$  is a basis of  $V$ .

(7) If vector space  $V$  has a basis  $B$  with four vectors, then  $\dim(V) = 4$ .

(T) This is how the dimension is defined.

(8) If  $S$  is a linearly independent set of a vector space  $V$ , then there exists a basis  $B$  that contains  $S$ .

(T) We can add more vectors into this set to become a basis.

(9) If  $S$  is a set vectors in a vector space  $V$ , and  $\text{span}(S) = V$ , then there exists a basis  $B$  that contains  $S$ .

(F) There exists a basis that is contained in  $S$ .

(10) If  $S = \{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3\}$  is a basis of a vector space  $V$ , then  $T = \{\mathbf{v}_1 - \mathbf{v}_2 - \mathbf{v}_3, -\mathbf{v}_1 + \mathbf{v}_2 - \mathbf{v}_3, -\mathbf{v}_1 - \mathbf{v}_2 + \mathbf{v}_3\}$  is also a basis of  $V$ .

(T) Because all vectors in  $T$  are linear combinations of vectors in  $S$ ,  $\text{span } T \subseteq \text{span } S$ . On the other hand, since  $\mathbf{v}_1 = -\frac{1}{2}(-\mathbf{v}_1 + \mathbf{v}_2 - \mathbf{v}_3) - \frac{1}{2}(-\mathbf{v}_1 - \mathbf{v}_2 + \mathbf{v}_3)$ ,  $\mathbf{v}_2 = -\frac{1}{2}(\mathbf{v}_1 - \mathbf{v}_2 - \mathbf{v}_3) - \frac{1}{2}(-\mathbf{v}_1 - \mathbf{v}_2 + \mathbf{v}_3)$ ,  $\mathbf{v}_3 = -\frac{1}{2}(-\mathbf{v}_1 + \mathbf{v}_2 - \mathbf{v}_3) - \frac{1}{2}(-\mathbf{v}_1 - \mathbf{v}_2 + \mathbf{v}_3)$ ,  $\text{span } S \subseteq \text{span } T$ . Hence,  $\text{span } T = \text{span } S = V$ . Since  $T$  spans  $V$  and  $|T| = 3 = \dim V$ ,  $T$  is a basis of  $V$ .

(11) If  $S = \{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3\}$  is a basis of a vector space  $V$ , then  $T = \{\mathbf{v}_1, \mathbf{v}_1 - \mathbf{v}_2 + \mathbf{v}_3, \mathbf{v}_1 + \mathbf{v}_2 - \mathbf{v}_3\}$  is also a basis of  $V$ .

(F) Since  $(\mathbf{v}_1 - \mathbf{v}_2 + \mathbf{v}_3) + (\mathbf{v}_1 + \mathbf{v}_2 - \mathbf{v}_3) = 2\mathbf{v}_1$ ,  $T$  is linearly dependent.

(12) If  $W$  is a subspace of a vector space  $V$ , then  $\dim(W) < \dim(V)$ .

(F) When  $W = V$ ,  $V$  is itself a subspace of  $V$ . In this case, we have  $\dim(W) = \dim(V)$ .

2. True-false questions about matrix multiplication and inversion:

Let  $A$ ,  $B$ , and  $C$  be  $n \times n$  matrices, and  $c$  is a nonzero scalar. Determine whether each of the following statements is always true:

(1)  $AB = BA$ .

(F) Matrix multiplication does not satisfy the commutative law.

(2) If the product  $AB$  is the zero matrix, then either  $A$  or  $B$  is a zero matrix.

(F) The product of two nonzero matrices may be a zero matrix.

(3) If  $AB = AC$ , then  $B = C$ .

(F) Matrix multiplication does not satisfy the cancelation law.

(4)  $(A + B)(A - B) = A^2 - B^2$ .

(F)  $(A + B)(A - B) = A^2 - AB - BA + B^2$ .

(5)  $c(A + B)^T = cA^T + cB^T$ .

(T)  $c(A + B)^T = c(A^T + B^T) = cA^T + cB^T$

(6)  $c(AB) = (cA)(cB)$ .

(F)  $c(AB) = (cA)B = A(cB)$ .

(7)  $(cA^{-1}) = cA^{-1}$ .

(T)

(8)  $(cA)^{-1} = cA^{-1}$ .

(F)  $(cA^{-1}) = c^{-1}A^{-1}$ .

(9)  $((AB)^T)^{-1} = (A^T)^{-1}(B^T)^{-1}$ .

(T)  $((AB)^T)^{-1} = (B^T A^T)^{-1} = (A^T)^{-1}(B^T)^{-1}$ .

3. Let  $\mathbf{F}$  be the set of all one-variable real-value functions defined on  $\mathbb{R}$ . Show that the subset  $S = \{p \in \mathbf{F} \mid p(1) = p(-1)\}$  is a subspace of  $\mathbf{F}$ .

*Solution.* The zero function  $p(x) = 0$  satisfies the condition  $p(1) = p(-1) = 0$ . Hence the zero function is in  $S$ .

If  $p(x) \in S$ , then  $(cp)(1) = c(p(1)) = c(p(-1)) = (cp)(-1)$ . Hence,  $cp \in S$ .

If  $p_1, p_2 \in S$ , then  $p_1(1) = p_1(-1)$  and  $p_2(1) = p_2(-1)$ . Hence,  $(p_1 + p_2)(1) = p_1(1) + p_2(1) = p_1(-1) + p_2(-1) = (p_1 + p_2)(-1)$ . Then  $p_1 + p_2 \in S$ .

$S$  is a subspace of  $\mathbf{F}$ .

4. Show that the set  $S = \{(x, y, z) \mid x + y = 2z\}$  is a subspace of  $\mathbb{R}^3$ .

*Solution.* When  $x = y = z = 0$ ,  $x + y = 0 = 2z$ . Hence,  $(0, 0, 0) \in S$ .

If  $(x, y, z) \in S$ , then  $x + y = 2z$ . Let  $x' = cx$ ,  $y' = cy$ , and  $z' = cz$ . Then  $c(x, y, z) = (x', y', z')$ . Since  $x' + y' = cx + cy = c(x + y) = c(2z) = 2(cz) = 2z'$ ,  $c(x, y, z) \in S$ .

If  $(x_1, y_1, z_1), (x_2, y_2, z_2) \in S$ , then  $x_1 + y_1 = 2z_1$  and  $x_2 + y_2 = 2z_2$ . Let  $x' = x_1 + x_2$ ,  $y' = y_1 + y_2$ , and  $z' = z_1 + z_2$ . Then  $(x_1, y_1, z_1) + (x_2, y_2, z_2) = (x', y', z')$ . Since  $x' + y' = x_1 + x_2 + y_1 + y_2 = (x_1 + y_1) + (x_2 + y_2) = 2z_1 + 2z_2 = 2(z_1 + z_2) = 2z'$ ,  $(x_1, y_1, z_1) + (x_2, y_2, z_2) \in S$ .

$S$  is a subspace of  $\mathbb{R}^3$ .

5. Show that  $S = \{(x, y) \mid x - y^2 = 0\}$  is not a subspace of  $\mathbb{R}^2$ .

*Solution.* Since  $4 = 2^2$ ,  $(4, 2) \in S$ .  $2(4, 2) = (8, 4)$ . Since  $8 \neq 4^2$ , the set  $S$  is not closed with respect to scalar multiplication.  $S$  is not a subspace of  $\mathbb{R}^2$ .

6. Show that  $S = \{(x, y) \mid x - y = 1\}$  is not a subspace of  $\mathbb{R}^2$ .

*Solution.* Since the zero vector  $(0, 0)$  is not in  $S$ ,  $S$  is not a subspace of  $\mathbb{R}^2$ .

7. For which value(s) of  $k$ , is the set  $S = \{(1, -2, 3), (1, 0, -1), (-1, 3, k)\}$  linearly independent?

Let  $x(1, -2, 3) + y(1, 0, -1) + z(-1, 3, k) = (0, 0, 0)$ . If this system has only the zero solution, then set  $S$  is linearly independent. Reduce the coefficient matrix of this system to an REF:

$$\begin{bmatrix} 1 & -2 & 3 \\ 1 & 0 & -1 \\ -1 & 3 & k \end{bmatrix} \rightarrow \begin{bmatrix} 1 & -2 & 3 \\ 0 & 2 & -4 \\ 0 & 1 & k+3 \end{bmatrix} \rightarrow \begin{bmatrix} 1 & -2 & 3 \\ 0 & 1 & -2 \\ 0 & 1 & k+3 \end{bmatrix} \rightarrow \begin{bmatrix} 1 & -2 & 3 \\ 0 & 1 & -2 \\ 0 & 0 & k+5 \end{bmatrix}.$$

When  $k \neq -5$ , this system has only the zero solution, and  $S$  is linearly independent.

8. Let  $\mathbf{P}_3$  be the vector space that consists of all polynomials of degree no more than 3. Show that  $S = \{1, 1 + x, x + x^2, x^2 + x^3\}$  is a basis of  $\mathbf{P}_3$ .

*Solution.* First we show that  $S$  is linearly independent.

Suppose  $c_1(1) + c_2(1 + x) + c_3(x + x^2) + c_4(x^2 + x^3) = 0$ . Then  $(c_1 + c_2) + (c_2 + c_3)x + (c_3 + c_4)x^2 + c_4x^3 = 0$ , and  $c_1 + c_2 = 0$ ,  $c_2 + c_3 = 0$ ,  $c_3 + c_4 = 0$ , and  $c_4 = 0$ . Hence,  $c_3 = c_2 = c_1 = 0$ . This set is linearly independent.

Next, we show that  $\text{span } S = \mathbf{P}_3$ . Let  $p = a + bx + cx^2 + dx^3 \in \mathbf{P}_3$ , and let

$$a_1(1) + a_2(1 + x) + a_3(x + x^2) + a_4(x^2 + x^3) = a + bx + cx^2 + dx^3. \text{ Then}$$

$$a_1 + a_2 = a,$$

$$a_2 + a_3 = b,$$

$$a_3 + a_4 = c,$$

$$a_4 = d.$$

Then  $a_3 = c - d$ ,  $a_2 = b - c$ ,  $a_1 = a - b$ . This system has a unique solution. Hence,  $\text{span } S = \mathbf{P}_3$ .

$S$  is a basis of  $\mathbf{P}_3$ .

You may also use the fact that  $\dim(\mathbf{P}_3) = 4$ . Since  $S$  has four vectors, you need to prove either  $S$  is independent, **or**  $\text{span}(S) = \mathbf{P}_3$ . Then combining the fact that the dimension of  $\mathbf{P}_3$  is 4,  $S$  is a basis of  $\mathbf{P}_3$ .

9. For which value(s) of  $a$ , is the vector  $(1, 4, 2)$  in  $\text{span}\{(a, 3, 1), (1, -4, -1)\}$ ?

*Solution.* Let  $x(a, 3, 1) + y(1, -4, -1) = (1, 4, 2)$ . Then  $ax + y = 1$ ,  $3x - 4y = 4$ ,  $x - y = 2$ .

If this system is consistent, then  $(1, 4, 2) \in \text{span}\{(a, 3, 1), (1, -4, -1)\}$ . Reduce the augmented matrix of this system to an REF:

$$\begin{aligned} \begin{bmatrix} a & 1 & 1 \\ 3 & -4 & 4 \\ 1 & -1 & 2 \end{bmatrix} &\xrightarrow{R_1 \leftrightarrow R_3} \begin{bmatrix} 1 & -1 & 2 \\ 3 & -4 & 4 \\ a & 1 & 1 \end{bmatrix} \xrightarrow{\substack{R_2 \rightarrow R_2 - 3R_1 \\ R_3 \rightarrow R_3 - aR_1}} \begin{bmatrix} 1 & -1 & 2 \\ 0 & -1 & -2 \\ 0 & 1+a & 1-2a \end{bmatrix} \xrightarrow{R_2 \rightarrow -R_2} \begin{bmatrix} 1 & -1 & 2 \\ 0 & 1 & 2 \\ 0 & 1+a & 1-2a \end{bmatrix} \\ &\xrightarrow{R_3 \rightarrow R_3 - (1+a)R_2} \begin{bmatrix} 1 & -1 & 2 \\ 0 & 1 & 2 \\ 0 & 0 & -1-4a \end{bmatrix}. \end{aligned}$$

This system is consistent if  $-1 - 4a = 0$ , or  $a = -\frac{1}{4}$ .

10. Find the inverse of the matrix  $A = \begin{bmatrix} 1 & -2 & 1 \\ 4 & -7 & 3 \\ -2 & 6 & -3 \end{bmatrix}$ .

$$\begin{aligned} \text{Solution. } & \begin{bmatrix} 1 & -2 & 1 & 1 & 0 & 0 \\ 4 & -7 & 3 & 0 & 1 & 0 \\ -2 & 6 & -3 & 0 & 0 & 1 \end{bmatrix} \xrightarrow{\substack{R_2 \rightarrow R_2 - 4R_1 \\ R_3 \rightarrow R_3 + 2R_1}} \begin{bmatrix} 1 & -2 & 1 & 1 & 0 & 0 \\ 0 & 1 & -1 & -4 & 1 & 0 \\ 0 & 2 & -1 & 2 & 0 & 1 \end{bmatrix} \xrightarrow{R_3 \rightarrow R_3 - 2R_2} \\ & \begin{bmatrix} 1 & -2 & 1 & 1 & 0 & 0 \\ 0 & 1 & -1 & -4 & 1 & 0 \\ 0 & 0 & 1 & 10 & -2 & 1 \end{bmatrix} \xrightarrow{R_1 \rightarrow R_1 + 2R_2} \begin{bmatrix} 1 & 0 & -1 & -7 & 2 & 0 \\ 0 & 1 & -1 & -4 & 1 & 0 \\ 0 & 0 & 1 & 10 & -2 & 1 \end{bmatrix} \xrightarrow{\substack{R_1 \rightarrow R_1 + R_3 \\ R_2 \rightarrow R_2 + R_3}} \begin{bmatrix} 1 & 0 & 0 & 3 & 0 & 1 \\ 0 & 1 & 0 & 6 & -1 & 1 \\ 0 & 0 & 1 & 10 & -2 & 1 \end{bmatrix} \end{aligned}$$

$A$  is invertible, and its inverse is  $\begin{bmatrix} 3 & 0 & 1 \\ 6 & -1 & 1 \\ 10 & -2 & 1 \end{bmatrix}$ .