1 Sequences and Series

EXAMPLE 1-1:

$$\lim_{n\to\infty} a_n = \lim_{n\to\infty}\frac{2}{n}$$
$$= 0$$
$$\therefore \text{convergent}$$

$$\lim_{n\to\infty} a_n = \lim_{n\to\infty}\frac{\ln n}{n}$$
$$= \lim_{n\to\infty}\frac{1}{n}$$
$$0$$
$$\therefore \text{convergent}$$

$$\lim_{n\to\infty} a_n = \lim_{n\to\infty}\frac{e^n}{n^3+4}$$
$$= \lim_{n\to\infty}\frac{e^n}{3n^2}$$
$$= \lim_{n\to\infty}\frac{e^n}{6n}$$
$$= \lim_{n\to\infty}\frac{e^n}{6}$$
$$= \infty$$
$$\therefore \text{divergent}$$

Squeeze Theorem.

$$\frac{-1}{n} \le \frac{\sin n}{n} \le \frac{1}{n}$$
$$\lim_{n\to\infty}\frac{-1}{n} = 0$$
$$\lim_{n\to\infty}\frac{1}{n} = 0$$
$$\lim_{n\to\infty}\frac{\sin n}{n} = 0$$
$$\therefore \text{convergent}$$

$$\lim_{n\to\infty} a_n = \lim_{n\to\infty} e^{2/n}$$
$$= e^0$$
$$= 1$$
$$\therefore \text{convergent}$$

EXAMPLE 1-2:

$$\sum_{n=0}^{\infty} 2\left(\frac{1}{2}\right)^{2n} = \sum_{n=0}^{\infty} 2\left[\left(\frac{1}{2}\right)^{2}\right]^{n}$$

$$= \sum_{n=0}^{\infty} 2\left(\frac{1}{4}\right)^{n}$$

$$r = \frac{1}{4} \rightarrow \text{convergent}$$

$$sum = \frac{a}{1-r} = \frac{2}{1-\frac{1}{4}} = \frac{2}{3/4} = 2\left(\frac{4}{3}\right) = \frac{8}{3}$$

EXAMPLE 1-3:

$$\sum_{n=1}^{\infty} 3\left(\frac{4}{3}\right)^{n+1} - \sum_{n=1}^{\infty} 2\left(\frac{1}{2}\right)^{n} = \sum_{n=1}^{\infty} 3\left(\frac{4}{3}\right)^{n-1}\left(\frac{4}{3}\right)^{2} - \sum_{n=1}^{\infty} 2\left(\frac{1}{2}\right)^{n-1}\left(\frac{1}{2}\right)^{1}$$

$$= \sum_{n=1}^{\infty}\left(\frac{16}{3}\right)\left(\frac{4}{3}\right)^{n-1} - \sum_{n=1}^{\infty}\left(\frac{1}{2}\right)^{n-1}$$

Series 1: $r = 4/3 \rightarrow$ divergent

Series 2: $r = 1/2 \rightarrow$ convergent

$sum = \infty$

EXAMPLE 1-4:

$$\sum_{n=2}^{\infty}\frac{3}{n^2+5n+6}=\sum_{n=2}^{\infty}\frac{3}{(n+2)(n+3)}$$

Partial Fractions.

$$\frac{3}{(n+2)(n+3)}=\frac{A}{n+2}+\frac{B}{n+3}$$

$$\begin{aligned}3&=A(n+3)+B(n+2)\\&=An+3A+Bn+2B\\&=n(A+B)+1(3A+2B)\end{aligned}$$

$$\begin{aligned}n&:A+B=0\\1&:3A+2B=3\end{aligned}$$

$$\begin{aligned}A&=3\\B&=-3\end{aligned}$$

$$\begin{aligned}\sum_{n=2}^{\infty}\frac{3}{(n+2)(n+3)}&=\sum_{n=2}^{\infty}\frac{3}{n+2}-\sum_{n=2}^{\infty}\frac{3}{n+3}\\&=\left(\frac{3}{4}+\frac{3}{5}+\frac{3}{6}+\frac{3}{7}+\cdots+\lim_{n\to\infty}\frac{3}{n+2}\right)-\left(\frac{3}{5}+\frac{3}{6}+\frac{3}{7}+\cdots+\lim_{n\to\infty}\frac{3}{n+3}\right)\\&=\left(\frac{3}{4}+\cancel{\frac{3}{5}+\frac{3}{6}+\frac{3}{7}+\cdots+\lim_{n\to\infty}\frac{3}{n+2}}\right)-\left(\cancel{\frac{3}{5}+\frac{3}{6}+\frac{3}{7}+\cdots+\lim_{n\to\infty}\frac{3}{n+3}}\right)\\&=\frac{3}{4}\end{aligned}$$

EXAMPLE 1-5:

$$\begin{aligned}\sum_{n=1}^{\infty}\arctan(n+1)-\sum_{n=1}^{\infty}\arctan n &= \left[\arctan 2+\arctan 3+\arctan 4+\cdots+\lim_{n\to\infty}\arctan n+\lim_{n\to\infty}\arctan(n+1)\right]\\ &\quad -\left[\arctan 1+\arctan 2+\arctan 3+\arctan 4+\cdots+\lim_{n\to\infty}\arctan n\right]\\ &= \left[\cancel{\arctan 2+\arctan 3+\arctan 4}+\cdots+\cancel{\lim_{n\to\infty}\arctan n}+\lim_{n\to\infty}\arctan(n+1)\right]\\ &\quad -\left[\arctan 1+\cancel{\arctan 2+\arctan 3+\arctan 4}+\cancel{\cdots+\lim_{n\to\infty}\arctan n}\right]\\ &= \lim_{n\to\infty}\arctan(n+1)-\arctan 1\\ &= \frac{\pi}{2}-\frac{\pi}{4}\\ &= \frac{\pi}{4}\end{aligned}$$

EXAMPLE 1-6:

First, we can simplify the general term: $a_n=\ln\left(e^{3n+1}\right)=3n+1$.

The *ln* was just there to scare us.

Then, using the Divergence Test, $\lim_{n\to\infty}a_n=\lim_{n\to\infty}(3n+1)=\infty\neq 0$. Therefore, the series diverges.

EXAMPLE 1-7:

We can apply the Integral Test. Then, whatever the integral does, the series does, too!

$$\int_2^\infty \frac{1}{x(\ln x)^3}\,dx = \lim_{b\to\infty}\int_2^b \frac{1}{x(\ln x)^3}\,dx$$

Substitution.

$$u = \ln x$$
$$\frac{du}{dx} = \frac{1}{x}$$
$$dx = x\,du$$

$$= \lim_{b\to\infty}\int \frac{1}{x(u^3)}\,x\,du$$
$$= \lim_{b\to\infty}\int \frac{1}{u^3}\,du$$
$$= \lim_{b\to\infty}\int u^{-3}\,du$$
$$= \lim_{b\to\infty}\frac{u^{-2}}{-2}$$
$$= \lim_{b\to\infty}\left(-\frac{1}{2u^2}\right)$$
$$= \lim_{b\to\infty}\left[-\frac{1}{2(\ln x)^2}\right]\Bigg|_2^b$$
$$= \lim_{b\to\infty}\left(-\frac{1}{2}\right)\left[\frac{1}{(\ln b)^2}-\frac{1}{(\ln 2)^2}\right]$$
$$= \frac{1}{2(\ln 2)^2}$$

Since the integral converges, the series converges, too.

EXAMPLE 1-8:

The math is yelling at us. It's time to use the Ratio Test.

$$a_n = \frac{4^n}{(n+1)!}$$

$$a_{n+1} = \frac{4^{n+1}}{(n+2)!}$$

$$\frac{a_{n+1}}{a_n} = \frac{4^{n+1}}{(n+2)!}\frac{(n+1)!}{4^n}$$

$$= \frac{4}{n+2}$$

$$\left|\frac{a_{n+1}}{a_n}\right| = \frac{4}{n+2}$$

$$\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty}\frac{4}{n+2} = 0 < 1$$

Therefore, the series converges.

EXAMPLE 1-9:

The series is to the power of n. Root Test!

We can determine a_n by taking the n^{th} root of the general term, which gives us $a_n = \frac{3n^2+2}{4n^2+5n}$.

Then,

$$\lim_{n\to\infty} a_n = \lim_{n\to\infty}\frac{3n^2+2}{4n^2+5n} = \frac{\infty}{\infty} \to \text{hospital!}$$

$$= \lim_{n\to\infty}\frac{6n}{8n+5} = \frac{\infty}{\infty} \to \text{hospital!}$$

$$= \lim_{n\to\infty}\frac{6}{8}$$

$$= \frac{3}{4} < 1$$

Therefore, the series converges.

EXAMPLE 1-10:

First, we will test the series *with* the alternator. We can use the Alternating Series Test to do this.

$$b_n = \frac{10}{n^2+1}$$

$$b_{n+1} = \frac{10}{(n+1)^2+1}$$

$$\lim_{n\to\infty} b_n = \lim_{n\to\infty} \frac{10}{n^2+1} = 0 \ \ (check!)$$

$$n+1 > n$$

$$(n+1)^2 > n^2$$

$$(n+1)^2+1 > n^2+1$$

$$\frac{1}{(n+1)^2+1} < \frac{1}{n^2+1}$$

$$\frac{10}{(n+1)^2+1} < \frac{10}{n^2+1}$$

$$b_{n+1} < b_n \ \ (check!)$$

The alternating series converges.

Now, let's remove the alternator, and see if the series *still* converges.

We now have $\sum_{n=1}^{\infty} \frac{10}{n^2+1}$. To test it, we need another test… We can't use the AST, but it's no longer an alternating series. The Integral Test will work very well. Let's try that!

$$\begin{aligned}\int_1^\infty \frac{10}{x^2+1}dx &= \lim_{b\to\infty}\int_1^b \frac{10}{x^2+1}dx \\ &= \lim_{b\to\infty}\left(10\arctan x\right)\Big|_1^b \\ &= \lim_{b\to\infty}10\left(\arctan b - \arctan 1\right) \\ &= 10\left(\frac{\pi}{2}-\frac{\pi}{4}\right) \\ &= 10\left(\frac{\pi}{4}\right) \\ &= \frac{5}{2}\pi\end{aligned}$$

The integral converges. Therefore, the series converges *without* the alternator.

Since the series converges ***with*** and ***without*** the alternator, we can conclude that the series is ***absolutely convergent***.

EXAMPLE 1-11:

$$\sum_{n=1}^{\infty} \frac{3\cos\left(\frac{n\pi}{2}\right)}{n^2+4} \cong \int_1^{\infty} \frac{3\cos\left(\frac{x\pi}{2}\right)}{x^2+4}\,dx$$ <<<< This is definitely Bob.

Numerator:

$$-1 \le \cos x \le 1 \qquad x:[1,\infty]$$

$$-1 \le \cos\left(\frac{x\pi}{2}\right) \le 1$$

$$-3 \le 3\cos\left(\frac{x\pi}{2}\right) \le 3$$

Denominator:

$$x^2+4 \ge x^2$$

$$\frac{1}{x^2+4} \le \frac{1}{x^2} \qquad x:[1,\infty]$$

Combination:

$$\underbrace{\frac{3\cos\left(\frac{x\pi}{2}\right)}{x^2+4}}_{Bob} \le \underbrace{\frac{3}{x^2}}_{Katrina} \qquad x:[1,\infty]$$

Based on the *p*-test, Katrina converges. Since Katrina is larger than Bob and converges, Bob must converge too.

Therefore, the original series converges.

__EXAMPLE 1-12:__

$$\sum_{n=1}^{\infty}\frac{3n^2+4}{n^3+2}$$

$$a_n=\frac{3n^2+4}{n^3+2} \qquad \text{(ugly)}$$

$$b_n=\frac{3n^2\cancel{+4}}{n^3\cancel{+2}}=\frac{3n^2}{n^3}=\frac{3}{n} \qquad \text{(simpler)}$$

$$\begin{aligned}\lim_{n\to\infty}\frac{a_n}{b_n}&=\lim_{n\to\infty}\frac{3n^2+4}{n^3+2}\frac{n}{3}\\ &=\lim_{n\to\infty}\frac{3n^3+4n}{3n^3+6}=\frac{\infty}{\infty}\to\text{hospital}\\ &=\lim_{n\to\infty}\frac{9n^2+4}{9n^2}=\frac{\infty}{\infty}\to\text{hospital}\\ &=\lim_{n\to\infty}\frac{18n}{18n}\\ &=\frac{18}{18}\\ &=1\\ &>0\end{aligned}$$

Since b_n diverges (based on p-test), a_n diverges too.

EXAMPLE 1-13:

a) Ratio Test.

$$a_n = \frac{4^n n^2}{n!}$$

$$a_{n+1} = \frac{4^{n+1}\left(n+1\right)^2}{\left(n+1\right)!}$$

$$\begin{aligned}\frac{a_{n+1}}{a_n} &= \frac{4^{n+1}\left(n+1\right)^2}{\left(n+1\right)!}\frac{n!}{4^n n^2}\\ &= \frac{4}{n+1}\frac{\left(n+1\right)^2}{n^2}\\ &= \frac{4\left(n+1\right)}{n^2}\\ &= \frac{4n+4}{n^2}\end{aligned}$$

$$\left|\frac{a_{n+1}}{a_n}\right| = \frac{4n+4}{n^2}$$

$$\begin{aligned}\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| &= \lim_{n\to\infty}\frac{4n+4}{n^2}\\ &= \lim_{n\to\infty}\frac{4}{2n}\\ &= 0 < 1 \to \text{convergent}\end{aligned}$$

b) Divergence Test.

$$\begin{aligned}\lim_{n\to\infty} a_n &= \lim_{n\to\infty}\frac{n}{e^{-n}}\\ &= \lim_{n\to\infty} ne^n\\ &= \infty\\ &\neq 0 \to \text{divergent}\end{aligned}$$

c) Comparison Test.

$$\sum_{n=1}^{\infty} \frac{\sin^2 n+2}{n^3+1} \cong \int_1^{\infty} \frac{\sin^2 x+2}{x^3+1}\,dx$$

$\int_1^{\infty} \frac{\sin^2 x+2}{x^3+1}\,dx$ <<<< This is definitely Bob.

Numerator: $0 \le \sin^2 x \le 1 \qquad x:[1,\infty]$

$2 \le \sin^2 x+2 \le 3$

Denominator: $x^3+1 \ge x^3$

$\frac{1}{x^3+1} \le \frac{1}{x^3} \qquad x:[1,\infty]$

Combination: $\underbrace{\frac{\sin^2 x+2}{x^3+1}}_{Bob} \le \underbrace{\frac{3}{x^3}}_{Katrina} \qquad x:[1,\infty]$

Based on the *p*-test, Katrina converges. Since Katrina is larger than Bob and converges, Bob must converge too.

Therefore, the original series converges.

d) Alternating Series Test.

$$b_n = \frac{1}{n}$$
$$b_{n+1} = \frac{1}{n+1}$$

$$n+1 > n$$
$$\frac{1}{n+1} < \frac{1}{n}$$
$$b_{n+1} < b_n$$

$$\begin{aligned}\lim_{n\to\infty} b_n &= \lim_{n\to\infty}\frac{1}{n}\\ &= 0\end{aligned}$$

$\therefore$ convergent

e) Integral Test.

$$\begin{aligned}\sum_{n=1}^{\infty}\frac{\ln n}{n} &\cong \int_1^{\infty}\frac{\ln x}{x}\,dx\\ &= \lim_{b\to\infty}\int_1^{b}\frac{\ln x}{x}\,dx\\ &= \lim_{b\to\infty}\frac{1}{2}(\ln x)^2\Big|_1^b\\ &= \lim_{b\to\infty}\frac{1}{2}\left[(\ln b)^2-(\ln 1)^2\right]\\ &= \infty\end{aligned}$$

$\therefore$ divergent

f) Divergence Test (easier!).

$$\begin{aligned}
\lim_{n\to\infty} a_n &= \lim_{n\to\infty} \frac{n^3+2}{\sqrt{2n+3}} = \frac{\infty}{\infty} \to \text{hospital} \\
&= \lim_{n\to\infty} \frac{3n^2}{\frac{1}{2}(2n+3)^{-1/2}(2)} \\
&= \lim_{n\to\infty} \frac{3n^2}{\cancel{\frac{1}{2}}(2n+3)^{-1/2}\cancel{(2)}} \\
&= \lim_{n\to\infty} (3n^2)(2n+3)^{1/2} \\
&= \infty \\
&\neq 0 \\
&\therefore \text{divergent}
\end{aligned}$$

Limit Comparison Test (more difficult, but still works).

$$\sum_{n=1}^{\infty} \frac{n^3+2}{\sqrt{2n+3}}$$

$$a_n = \frac{n^3+2}{\sqrt{2n+3}} \qquad \text{(ugly)}$$

$$b_n = \frac{n^3 \cancel{+2}}{\sqrt{2n \cancel{+3}}} = \frac{n^3}{\sqrt{2n}} = \frac{n^3}{\sqrt{2}n^{1/2}} = \frac{n^{5/2}}{\sqrt{2}} \qquad \text{(simpler)}$$

$$\begin{aligned}
\lim_{n\to\infty} \frac{a_n}{b_n} &= \lim_{n\to\infty} \frac{n^3+2}{\sqrt{2n+3}} \frac{\sqrt{2}}{n^{5/2}} \\
&= \lim_{n\to\infty} \frac{n^3\left(1+\frac{2}{n^3}\right)}{n^{1/2}\sqrt{2+\frac{3}{n}}} \frac{\sqrt{2}}{n^{5/2}} \\
&= \lim_{n\to\infty} \frac{\sqrt{2}\,\cancel{n^3}\left(1+\frac{2}{n^3}\right)}{\cancel{n^3}\sqrt{2+\frac{3}{n}}} \\
&= \lim_{n\to\infty} \frac{\sqrt{2}\left(1+\cancel{\frac{2}{n^3}}\right)}{\sqrt{2+\cancel{\frac{3}{n}}}} \\
&= \frac{\sqrt{2}}{\sqrt{2}} \\
&= 1 \\
&> 0
\end{aligned}$$

Since b_n diverges (based on p-test), a_n diverges too.

EXAMPLE 1-14:

First, we will do the Alternating Series Test to see if the series converges *with* the alternator.

Alternating Series Test:

$$b_n = \frac{1}{n^4+2}$$
$$b_{n+1} = \frac{1}{(n+1)^4+2}$$

$$(n+1)^4 > n^4$$
$$(n+1)^4+2 > n^4+2$$
$$\frac{1}{(n+1)^4+2} < \frac{1}{n^4+2}$$
$$b_{n+1} < b_n$$

$$\lim_{n\to\infty} b_n = \lim_{n\to\infty}\frac{1}{n^4+2} = 0$$

$\therefore$ convergent

Now we will remove the alternator to see if the series still converges: $\sum_{n=1}^{\infty}\frac{1}{n^4+2}$. We can use the Comparison Test to test this series.

$$\sum_{n=1}^{\infty}\frac{1}{n^4+2} \cong \int_1^{\infty}\frac{1}{x^4+2}\,dx$$

$\int_1^{\infty}\frac{1}{x^4+2}\,dx$ <<<< This is definitely Bob.

$$x^4+2 \geq x^4$$
$$\underbrace{\frac{1}{x^4+2}}_{Bob} \leq \underbrace{\frac{1}{x^4}}_{Katrina} \qquad x:[1,\infty]$$

Based on the *p*-test, Katrina converges. Since Katrina is larger than Bob and converges, Bob must converge too.

Therefore, the original series converges.

Since both the series *with* and *without* the alternator converge, we can conclude that the series is *absolutely convergent*.

EXAMPLE 1-15:

Integral Test Error Estimation.

$$\begin{aligned} error &\le \int_{10}^{\infty} \frac{1}{x^3}\,dx \\ &\le \lim_{b\to\infty} \int_{10}^{b} \frac{1}{x^3}\,dx \\ &\le \lim_{b\to\infty} \left(-\frac{1}{2x^2}\right)\Bigg|_{10}^{b} \\ &\le \lim_{b\to\infty} \left[-\frac{1}{2b^2} + \frac{1}{2(10)^2}\right] \\ &\le \frac{1}{200} \end{aligned}$$

Alternating Series Test Error Estimation.

$$\begin{aligned} error &\le b_{n+1} \\ &\le b_{11} \\ &\le \frac{3}{11^2 + 2(11)} \\ &\le \frac{3}{121+22} \\ &\le \frac{3}{143} \end{aligned}$$

EXAMPLE 1-16:

$$error \le \int_n^\infty \frac{1}{x^2}\,dx$$

$$0.001 \le \lim_{b\to\infty}\int_n^b \frac{1}{x^2}\,dx$$

$$0.001 \le \lim_{b\to\infty}\left(-\frac{1}{x}\right)\Bigg|_n^b$$

$$0.001 \le \lim_{b\to\infty}\left[-\frac{1}{b}+\frac{1}{n}\right]$$

$$0.001 \le \frac{1}{n}$$

$$n \ge \frac{1}{0.001}$$

$$n \ge 1000$$

1.6 Sequences and Series: Your Turn!

Solutions:

a)

i)

$$\begin{aligned}\lim_{n\to\infty} a_n &= \lim_{n\to\infty}\frac{3}{n^3}\\ &= 0\\ &\therefore \text{convergent}\end{aligned}$$

ii)

$$\begin{aligned}\lim_{n\to\infty} a_n &= \lim_{n\to\infty}\frac{2+\ln n}{n^2}\\ &= \lim_{n\to\infty}\frac{\frac{1}{n}}{2n}\\ &= \lim_{n\to\infty}\frac{1}{2n^2}\\ &= 0\\ &\therefore \text{convergent}\end{aligned}$$

iii)

$$\begin{aligned}\lim_{n\to\infty} a_n &= \lim_{n\to\infty}\frac{\left(e^{3n}+2\right)^2}{e^n}\\ &= \lim_{n\to\infty}\frac{2\left(e^{3n}+2\right)\left(3e^{3n}\right)}{e^n}\\ &= \lim_{n\to\infty}\frac{6e^{3n}\left(e^{3n}+2\right)}{e^n}\\ &= \lim_{n\to\infty}6e^{3n}e^{-n}\left(e^{3n}+2\right)\\ &= \lim_{n\to\infty}6e^{2n}\left(e^{3n}+2\right)\\ &= \infty\\ &\therefore \text{divergent}\end{aligned}$$

iv)

$$
\begin{aligned}
\ln a_n &= \ln\left[3e^{-4/\left(3n^3\right)}\right] \\
&= \ln 3 - \frac{4}{3n^3}
\end{aligned}
$$

$$
\begin{aligned}
\lim_{n\to\infty}\left(\ln a_n\right) &= \lim_{n\to\infty}\left(\ln 3 - \frac{4}{3n^3}\right) \\
&= \ln 3 - 0 \\
&= \ln 3 \\
\lim_{n\to\infty} a_n &= \lim_{n\to\infty} e^{\ln 3} \\
&= 3 \\
&\therefore \text{convergent}
\end{aligned}
$$

v)

$$
\begin{aligned}
\frac{-2}{n^4} \le \frac{\sin n - \cos n}{n^4} &\le \frac{2}{n^4} \\
\lim_{n\to\infty}\frac{-2}{n^4} &= 0 \\
\lim_{n\to\infty}\frac{2}{n^4} &= 0 \\
\lim_{n\to\infty}\frac{\sin n - \cos n}{n^4} &= 0 \quad \left(\text{Squeeze Theorem}\right) \\
&\therefore \text{convergent}
\end{aligned}
$$

b)

i)

$$\begin{aligned}
\sum_{n=1}^{\infty}\left(\frac{3}{2}\right)\left(\frac{2}{3}\right)^{2n+1}-\sum_{n=1}^{\infty}\left(\frac{1}{3}\right)\left(\frac{1}{3}\right)^{n+1} &= \sum_{n=1}^{\infty}\left(\frac{3}{2}\right)\left(\frac{2}{3}\right)^{2n}\left(\frac{2}{3}\right)^{1}-\sum_{n=1}^{\infty}\left(\frac{1}{3}\right)\left(\frac{1}{3}\right)^{n}\left(\frac{1}{3}\right)^{1} \\
&= \sum_{n=1}^{\infty}\left(\frac{3}{2}\right)\left[\left(\frac{2}{3}\right)^{2}\right]^{n}\left(\frac{2}{3}\right)-\sum_{n=1}^{\infty}\left(\frac{1}{3}\right)\left(\frac{1}{3}\right)^{n}\left(\frac{1}{3}\right) \\
&= \sum_{n=1}^{\infty}(1)\left(\frac{4}{9}\right)^{n}-\sum_{n=1}^{\infty}\left(\frac{1}{9}\right)\left(\frac{1}{3}\right)^{n} \\
&= \sum_{n=1}^{\infty}(1)\left(\frac{4}{9}\right)^{n-1}\left(\frac{4}{9}\right)^{1}-\sum_{n=1}^{\infty}\left(\frac{1}{9}\right)\left(\frac{1}{3}\right)^{n-1}\left(\frac{1}{3}\right)^{1} \\
&= \sum_{n=1}^{\infty}\left(\frac{4}{9}\right)\left(\frac{4}{9}\right)^{n-1}-\sum_{n=1}^{\infty}\left(\frac{1}{27}\right)\left(\frac{1}{3}\right)^{n-1}
\end{aligned}$$

Series 1: $r = 4/9 \rightarrow$ convergent

Series 2: $r = 1/3 \rightarrow$ convergent

$$\begin{aligned}
sum &= sum_1 - sum_2 \\
&= \frac{\frac{4}{9}}{1-\frac{4}{9}}-\frac{\frac{1}{27}}{1-\frac{1}{3}} \\
&= \frac{4/9}{5/9}-\frac{1/27}{2/3} \\
&= \frac{4}{5}-\frac{1}{2}\frac{3}{27} \\
&= \frac{4}{5}-\frac{1}{18} \\
&= \frac{72}{90}-\frac{5}{90} \\
&= \frac{67}{90}
\end{aligned}$$

ii)

$$\sum_{n=2}^{\infty}\frac{5}{n^2+7n+12}=\sum_{n=2}^{\infty}\frac{5}{(n+3)(n+4)}$$

Partial Fractions.

$$\frac{5}{(n+3)(n+4)}=\frac{A}{n+3}+\frac{B}{n+4}$$

$$\begin{aligned}5&=A(n+4)+B(n+3)\\&=An+4A+Bn+3B\\&=n(A+B)+1(4A+3B)\end{aligned}$$

$$\begin{aligned}n&:A+B=0\\1&:4A+3B=5\end{aligned}$$

$$\begin{aligned}A&=5\\B&=-5\end{aligned}$$

$$\begin{aligned}\sum_{n=2}^{\infty}\frac{5}{(n+3)(n+4)}&=\sum_{n=2}^{\infty}\frac{5}{n+3}-\sum_{n=2}^{\infty}\frac{5}{n+4}\\&=\left(\frac{5}{5}+\frac{5}{6}+\frac{5}{7}+\frac{5}{8}+\cdots+\lim_{n\to\infty}\frac{5}{n+3}\right)-\left(\frac{5}{6}+\frac{5}{7}+\frac{5}{8}+\cdots+\lim_{n\to\infty}\frac{5}{n+4}\right)\\&=\left(\frac{5}{5}+\cancel{\frac{5}{6}+\frac{5}{7}+\frac{5}{8}+\cdots+\lim_{n\to\infty}\frac{5}{n+3}}\right)-\left(\cancel{\frac{5}{6}+\frac{5}{7}+\frac{5}{8}+\cdots+\lim_{n\to\infty}\frac{5}{n+4}}\right)\\&=1\end{aligned}$$

iii)

$$\sum_{n=1}^{\infty}\arctan\left(n+3\right)-\sum_{n=1}^{\infty}\arctan\left(n+2\right)$$

$$=\left[\arctan 4+\arctan 5+\arctan 6+\cdots+\lim_{n\to\infty}\arctan\left(n+2\right)+\lim_{n\to\infty}\arctan\left(n+3\right)\right]$$

$$-\left[\arctan 3+\arctan 4+\arctan 5+\arctan 6+\cdots+\lim_{n\to\infty}\arctan\left(n+2\right)\right]$$

$$=\left[\cancel{\arctan 4+\arctan 5+\arctan 6}+\cdots+\cancel{\lim_{n\to\infty}\arctan\left(n+2\right)}+\lim_{n\to\infty}\arctan\left(n+3\right)\right]$$

$$-\left[\arctan 3+\cancel{\arctan 4+\arctan 5+\arctan 6}+\cdots+\cancel{\lim_{n\to\infty}\arctan\left(n+2\right)}\right]$$

$$=\lim_{n\to\infty}\arctan\left(n+3\right)-\arctan 3$$

$$=\frac{\pi}{2}-\arctan 3$$

c)

i) Comparison Test.

$$\sum_{n=1}^{\infty}\frac{1+\cos n}{n^4}\cong\int_1^{\infty}\frac{1+\cos x}{x^4}\,dx$$

$\int_1^{\infty}\frac{1+\cos x}{x^4}\,dx$ <<<< This is definitely Bob.

$$-1\le\cos x\le 1 \qquad\qquad x:\left[1,\infty\right]$$

$$0\le 1+\cos x\le 2$$

$$\underbrace{0}_{Brian}\le\underbrace{\frac{1+\cos x}{x^4}}_{Bob}\le\underbrace{\frac{2}{x^4}}_{Katrina}$$

Based on the *p*-test, Katrina converges. Since Katrina is larger than Bob and converges, Bob must converge too.

Therefore, the original series converges.

ii) Alternating Series Test.

$$b_n = \frac{1}{n+2}$$

$$b_{n+1} = \frac{1}{n+3}$$

$$\begin{aligned} n+3 &> n+2 \\ \frac{1}{n+3} &< \frac{1}{n+2} \\ b_{n+1} &< b_n \end{aligned}$$

$$\begin{aligned} \lim_{n\to\infty} b_n &= \lim_{n\to\infty} \frac{1}{n+2} \\ &= 0 \end{aligned}$$

$\therefore$ convergent

iii) Ratio Test.

$$a_n = \frac{3^n\left(n+1\right)^2}{\left(n+1\right)!}$$

$$a_{n+1} = \frac{3^{n+1}\left(n+2\right)^2}{\left(n+2\right)!}$$

$$\frac{a_{n+1}}{a_n} = \frac{3^{n+1}\left(n+2\right)^2}{\left(n+2\right)!}\frac{\left(n+1\right)!}{3^n\left(n+1\right)^2}$$

$$= \frac{3}{n+2}\frac{\left(n+2\right)^2}{\left(n+1\right)^2}$$

$$= \frac{4\left(n+2\right)}{\left(n+1\right)^2}$$

$$= \frac{4n+8}{\left(n+1\right)^2}$$

$$\left|\frac{a_{n+1}}{a_n}\right| = \frac{4n+8}{\left(n+1\right)^2}$$

$$\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty}\frac{4n+8}{\left(n+1\right)^2}$$

$$= \lim_{n\to\infty}\frac{4}{2\left(n+1\right)}$$

$$= 0 < 1 \rightarrow \text{convergent}$$

iv) Divergence Test.

$$\begin{aligned}
\lim_{n\to\infty} a_n &= \lim_{n\to\infty}\frac{3ne^n}{n^4} \\
&= \lim_{n\to\infty}\frac{3e^n}{n^3} \\
&= \lim_{n\to\infty}\frac{3e^n}{3n^2} \\
&= \lim_{n\to\infty}\frac{3e^n}{6n} \\
&= \lim_{n\to\infty}\frac{3e^n}{6} \\
&= \infty \\
&\neq 0 \rightarrow \text{divergent}
\end{aligned}$$

v) Integral Test.

$$\begin{aligned}
\sum_{n=1}^{\infty}\frac{3}{n^2+1} &\cong \int_1^{\infty}\frac{3}{x^2+1}\,dx \\
&= \lim_{b\to\infty}\int_1^{b}\frac{3}{x^2+1}\,dx \\
&= \lim_{b\to\infty} 3\arctan x\Big|_1^b \\
&= \lim_{b\to\infty} 3\left(\arctan b - \arctan 1\right) \\
&= 3\left(\frac{\pi}{2}-\frac{\pi}{4}\right) \\
&= \frac{3\pi}{4} \\
&\therefore \text{convergent}
\end{aligned}$$

vi) Limit Comparison Test.

$$\sum_{n=0}^{\infty}\frac{1}{4^n+2}$$

$$a_n = \frac{1}{4^n+2} \qquad \text{(ugly)}$$

$$b_n = \frac{1}{4^n \cancel{+2}} = \frac{1}{4^n} \qquad \text{(simpler)}$$

$$\lim_{n\to\infty}\frac{a_n}{b_n} = \lim_{n\to\infty}\frac{1}{4^n+2}\frac{4^n}{1} \to \text{can't simplify ratio} \quad \therefore \text{ try flipping it}$$

$$\lim_{n\to\infty}\frac{b_n}{a_n} = \lim_{n\to\infty}\frac{1}{4^n}\frac{4^n+2}{1}$$

$$= \lim_{n\to\infty}\left(\frac{4^n}{4^n}+\frac{2}{4^n}\right)$$

$$= \lim_{n\to\infty}\left(1+\frac{2}{4^n}\right)$$

$$= \lim_{n\to\infty}\left(1+\cancel{\frac{2}{4^n}}\right)$$

$$= 1$$

$$> 0$$

Since b_n converges (it is a geometric series with r = ¼), a_n converges too.

d) Alternating Series Test:

$$b_n = \frac{n}{n^2+1}$$
$$b_{n+1} = \frac{n+1}{(n+1)^2+1}$$

$$(n+1)^2 > n^2$$
$$(n+1)^2+1 > n^2+1$$
$$\frac{1}{(n+1)^2+1} < \frac{1}{n^2+1}$$
$$\frac{n+1}{(n+1)^2+1} < \frac{n}{n^2+1}$$
$$b_{n+1} < b_n$$

$$\begin{aligned}\lim_{n\to\infty} b_n &= \lim_{n\to\infty}\frac{n}{n^2+1}\\ &= \lim_{n\to\infty}\frac{1}{2n}\\ &= 0\\ &\therefore \text{convergent}\end{aligned}$$

Remove alternator, and test series again:

Integral Test.

$$\begin{aligned}\sum_{n=1}^{\infty}\frac{n}{n^2+1} &\cong \int_1^{\infty}\frac{x}{x^2+1}dx\\ &= \lim_{b\to\infty}\underbrace{\int_1^b \frac{x}{x^2+1}dx}_{u=x^2+1}\\ &= \lim_{b\to\infty}\frac{1}{2}\ln\left(x^2+1\right)\Big|_1^b\\ &= \lim_{b\to\infty}\frac{1}{2}\left[\ln\left(b^2+1\right)-\ln\left(1^2+1\right)\right]\\ &= \infty\\ &\therefore \text{divergent}\end{aligned}$$

Since the series converges *with* the alternator and diverges *without* the alternator, the series overall is *conditionally convergent.*

e)

i) *Integral Test Error Estimation.*

$$
\begin{aligned}
error &\le \int_6^\infty \frac{3}{x^4}\,dx \\
&\le \lim_{b\to\infty}\int_6^b \frac{3}{x^4}\,dx \\
&\le \lim_{b\to\infty}\left(-\frac{3}{3x^3}\right)\Bigg|_6^b \\
&\le \lim_{b\to\infty}\left[-\frac{1}{b^3}+\frac{1}{6^3}\right] \\
&\le \frac{1}{6^3}
\end{aligned}
$$

ii) *Alternating Series Test Error Estimation.*

$$
\begin{aligned}
error &\le b_{n+1} \\
&\le b_7 \\
&\le \frac{3+7}{6(7)^2+(7)^3} \\
&\le \frac{10}{6(49)+7^3} \\
&\le \frac{10}{294+343} \\
&\le \frac{10}{637}
\end{aligned}
$$