Assignment II
Due date: 17 October 2018
Note: You have to drop off your assignment in the class. It is advised that you make a copy of your assignment.
Note: I will post solutions to all questions. However, only selected questions will be marked.

Q1. The probability mass function of X , the number of imperfections per 10 meters of a synthetic fabric in continuous rolls of uniform width, is given by

$$f(x) = \begin{cases} c(x^3+1)\,, & \text{for } x = 0, 1, 2, 3; \\ 0, & \text{elsewhere}\,. \end{cases}$$

Determine the following quantities:

(a)	c	(b)	$P(X=1)$	(c)	$P(0.5 < X < 2.7)$
(d)	$P(X > 3)$	(e)	$P(0 \le X < 2)$	(f)	Mean of X
(g)	Variance of X	(h)	Distribution function of X		

Solution to Q1:

Since $f(x)$ is a probability mass function we should have is

$$\sum_{i=0}^{3} f(i) = c40 = 1\,.$$

Therefore
(a) $c = 1/40\,.$

(b) $P(X=1) = \frac{1}{20}$

(c) $P(0.5 < X < 2.7) = P(X=1) + P(X=2) = \frac{11}{40}$

(d) $P(X > 3) = 0$

(e) $P(0 \le X < 2) = P(X=0) + P(X=1) = \frac{3}{40}$

(f) $\mathrm{E}(X) = \sum_{i=0}^{3} \frac{1}{40} i(1+i^3) = \frac{104}{40}$

(g) $\mathrm{E}(X^2) = \sum_{i=0}^{3} \frac{1}{40} i^2(1+i^3) = \frac{290}{40}$, Thus $\sigma^2 = \mathrm{E}(X^2) - (\mathrm{E}(X))^2 = 0.49\,.$

(h)

$$F(x) = \begin{cases} 0 & x < 0 \\ \frac{1}{40} & 0 \le x < 1 \\ \frac{3}{40} & 1 \le x < 2 \\ \frac{12}{40} & 2 \le x < 3 \\ 1 & x \ge 3 \end{cases}$$

Marking scheme for Q1:

For each part - 1 point. Total - 8 points.

Q2. On a laboratory assignment, if the equipment is working, the density function of the observed outcomes, X, is

$$f(x) = \begin{cases} c(x+0.5)\,, & \text{for } 0 < x < 1 \\ 0, & \text{elsewhere}\,. \end{cases}$$

Determine the following quantities:

(a) c (b) $P(X = 0.5)$ (c) $P(X < 0.75)$

(d) $P(X \geq 1.21)$ (e) Mean of X (f) Variance of X

(g) Distribution function of X

Solution to Q2:

Since $f(x)$ is a probability density function we should have is

$$\int f(x)dx = c\int_0^1 (x+0.5)dx = 1\,.$$

Therefore

(a) $c = 1\,.$

(b) $P(X = 0.5) = 0$

(c) $P(X < 0.75) = \int_0^{0.75} (x+0.5)dx = 0.65625$

(d) $P(X \geq 3) = 0$

(e) $\mathrm{E}(X) = \int_0^1 x(x+0.5)dx = 7/12$

(f) $\mathrm{E}(X^2) = \int_0^1 x^2(x+0.5)dx = 5/12\,,$ Thus $\sigma^2 = \mathrm{E}(X^2) - (\mathrm{E}(X))^2 = 0.0763\,.$

(g) $F(x) = \begin{cases} 0 & x < 0 \\ \frac{1}{2}\left(x^2 + x\right) & 0 \leq x < 1 \\ 1 & x \geq 1 \end{cases}$

Marking scheme for Q2:

This question will not be marked

Q3. The proportion of the budget for a certain type of industrial company that is allotted to environmental and pollution control is coming under security. A data collection project determines that the cumulative distribution function of these proportions is given by

$$F(x) = \begin{cases} 0 & x < 0\,, \\ x^2\,, & 0 \leq x < 1\,, \\ 1, & x \geq 1\,. \end{cases}$$

Determine the following quantities:

(a) $P(X < 0.5)$ (b) $P(X > 0.75)$ (c) $P(X \leq 1.21)$

(d) density function of X (e) Mean of X (f) Variance of X

Solution to Q3:

Since $F(x)$ is a cumulative distribution function

(a) $P(X < 0.5) = F(0.5) = 0.25$

(b) $P(X > 0.75) = 1 - F(0.75) = 0.4375$

(c) $P(X < 1.21) = F(1.21) = 1$

(d) Since $f(x) = \frac{d}{dx}F(x)$, $f(x) = \begin{cases} 0 & x < 0 \\ 2x & 0 \leq x < 1 \\ 1 & x \geq 1 \end{cases}$

(e) $\mathrm{E}(X) = \int_0^1 x(2x)dx = 2/3$

(f) $\mathrm{E}(X^2) = \int_0^1 x^2(2x)dx = 0.5$, Thus $\sigma^2 = \mathrm{E}(X^2) - (\mathrm{E}(X))^2 = 0.0555$.

Marking scheme for Q3:

For each part - 1 point. Total - 6 points.

Q4. Suppose that small aircrafts arrive at a certain airport, according to a poisson process, at the rate of 1 per day.

(a) What is the probability that 4 small aircrafts arrive during a two-days period?

(b) What is the probability that no small aircraft arrives during a 1-day period?

(c) What is the probability that in exactly four days of a week no small aircraft arrives?

(d) In how many days of a month we should expect that small aircrafts will arrive? (assume that the month has 30 days)

Solution to Q4:

We know that the parameter of the poisson process is $\lambda = 1$. Let X be the number of small craft that will be landed.

(a) $\lambda = 1$ and $t = 2$, $P(X = 4) = \frac{e^{-1*2}(1*2)^4}{4!} = 0.0902$.

(b) $\lambda = 1$ and $t = 1$, $P(X = 0) = \frac{e^{-1*1}(1*1)^0}{0!} = 0.3678$.

(c) If Y be the number of days without small aircraft arrival, Y is binomial random variable with parameters $n = 7$ and $p = 0.3678$(it calculated in the previous part), then $P(Y = 4) = \binom{7}{4} p^4(1-p)^3 = 0.1618$.

(d) If Y be the number of days with small aircraft arrival, Y is binomial random variable with parameters $n = 30$ and $p = 1-0.3678 = 0.6322$(it calculated in the previous part), $E(Y) = np = 30*0.6322 = 18.966$.

Marking scheme for Q4:

This question will not be marked

Q5. In a clinical study, volunteers are tested for a gene that has been found to increase the risk for a disease. The probability that the person carries the gene is 0.1.

(a) What is the probability that 4 or more people will have to be tested in order to detect one person with the gene?

(b) How many people are expected to be tested in order to detect one person with the gene?

(c) How many people are expected to be tested before 2 with the gene are detected?

Solution to Q5:

(a) If X is the number of steps before the 1st success, then X has geometric distribution with $p = 0.1$ (success = gene detection). To compute

$$P(X \geq 4) = \sum_{k=4}^{\infty}(1-p)^{k-1}p = p \times \frac{(1-p)^3}{1-(1-p)} = (1-p)^3 = 0.729.$$

or $P(X \geq 4) = 1 - P(X < 4) = 0.729$

(b) $\mathrm{E}(X) = 1/p = 10$.

(c) $\mathrm{E}(X) = 20$.

Marking scheme for Q5:

For each part - 1 point. Total - 3 points.

Q6. The number of failures of a testing instrument from contamination particles on the product is a Poisson random variable with a mean of 0.02 failures per hour.

(a) What is the probability that the instrument does not fail in an 8-hour shift?

(b) What is the probability that the first failure of the testing instrument does not happen in an 8-hour shift?

(c) What is the probability that the third failure of the testing instrument will happen in an 8-hour shift?

Solution to Q6:

(a) $\lambda = 0.02$ and $t = 8$, therefore, the failure rate per 8 hours is $8 \times 0.02 = 0.16$. If X is Poisson random variable with $\lambda = 0.16$, we have to compute $P(X = 0) = \exp(-0.16) = 0.8521438$.

(b) If Y is the time to the first failure it is an exponential random variable with parameter $\beta = 1/\lambda = 1/0.02$, and density function $f(x;\beta) = \frac{1}{\beta}e^{-x/\beta}$, when$x > 0$. Thus $P(Y > 8) = \int_8^\infty f(x;\beta)dx = \int_8^\infty 0.02e^{-0.02x}dx = \exp(-0.16) = 0.8521438$. (Note, that the first and second part have the same solution)

(c) If Y is the time to the third failure it is an exponential random variable with parameters $\alpha = 3$ and $\beta = 1/\lambda = 1/0.02$, and density function $f(x;\alpha,\beta) = \frac{1}{\Gamma(\alpha)\beta^\alpha}x^{\alpha-1}e^{-x/\beta}$, when$x > 0$. Thus $P(Y < 8) = \int_0^8 f(x;\alpha,\beta)dx = \int_0^8 \frac{0.02^3}{2!}x^2e^{-0.02x}dx$.

Marking scheme for Q6:

This question will not be marked

Q7. The tensile strength of a certain metal component is normally distributed with a mean of 10000 kilometers per square centimeter and a standard deviation of 100 kilograms per square centimeter.

(a) What proportion of these components are less than 10020 kilograms per square centimeter in tensile strength?

(b) What proportion of these components are between 9950 and 10100 kilograms per square centimeter in tensile strength?

(c) If specifications require that all components have tensile strength between 9900 and k kilograms per square centimeter inclusive and 79.77 percent of the proportion of pieces would we expect to scrap, determine the value of k.

Solution to Q7:

(a) $P(X < 10200) = P(Z < (10020 - 1000)/100) = 0.5792$

(b) $P(9950 < X < 10100) = P((9950 - 10000)/100 < Z < (10100 - 10000)/100) = 0.5328$

(c) $1 - 0.7977 = P(9900 < X < k) = P(Z < (k - 10000)/100) - P(Z < (9900 - 10000)/100) = P(Z < (k - 10000)/100) - P(Z < -1) = P(Z < (k - 10000)/100) - 0.1587$, therefore $P(Z < (k - 10000)/100) =$

$1 - 0.7977 + 0.1587 = 0.361$. The closet number to 0.361 in the table is 0.33594 which is corresponding to $P(Z < -0.36)$. Therefore, $(k - 10000)/100 = -0.36$ thus $k = 9964$.

Marking scheme for Q7:

This question will not be marked