# Lecture 03 (Reduced Echelon Forms)

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A matrix is in <u>reduced echelon form</u> if it is in echelon form and

REF (RREF).

4) The leading entries are 1.

5) All the numbers above a leading entry are zero

<u>Example</u>

$$\begin{pmatrix} 1 & 0 & 0 & 4 \\ 0 & 1 & 0 & 7 \\ 0 & 0 & 1 & 9 \end{pmatrix}$$ is in REF.

$$\begin{pmatrix} 1 & 2 & 0 & 4 \\ 0 & 0 & 1 & 5 \end{pmatrix}$$ is in REF.

<u>Example</u>

$$\begin{pmatrix} 1 & 2 & 3 & 4 \\ 0 & 0 & 1 & 5 \end{pmatrix}.$$ in echelon form but not in REF. (violates (5))

$$\begin{pmatrix} 1 & 2 & 3 & 4 \\ 0 & 0 & 2 & 4 \end{pmatrix}$$ " " " " " (violates (4)(5))

<u>Exercise</u>

$$\begin{pmatrix} * & * & * & * \\ * & * & * & * \\ * & * & * & * \end{pmatrix}$$

$$\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{pmatrix}$$

$$\begin{pmatrix} * & * & * & * \\ * & * & * & * \end{pmatrix} \qquad \begin{pmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{pmatrix}$$

$$\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 2 & 0 & 0 \\ 0 & 0 & 3 & 0 \end{pmatrix}$$

$$\begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 1 & 2 & 3 \\ 0 & 0 & 0 & 0 \end{pmatrix}.$$

**Big Picture:**

Echelon form $\longrightarrow$ Consistency (has a solution or not)

If consistent, number of solutions

Reduced Echelon form $\longrightarrow$ How to describe the solutions.

<u>Example</u> Reduce the matrix to an echelon form

$$\begin{pmatrix} 0 & -3 & -6 & 4 & 9 \\ -1 & -2 & -1 & 3 & 1 \\ -2 & -3 & 0 & 3 & -1 \\ 1 & 4 & 5 & -9 & -7 \end{pmatrix}.$$

<u>Sol.</u>

$$\xrightarrow[R1 \leftrightarrow R4]{} \begin{pmatrix} 1 & 4 & 5 & -9 & -7 \\ -1 & -2 & -1 & 3 & 1 \\ -2 & -3 & 0 & 3 & -1 \\ 0 & -3 & -6 & 4 & 9 \end{pmatrix}$$

$$\xrightarrow{R1 \leftrightarrow R4} \begin{pmatrix} \vdots \\ -2 & -3 & 0 & 3 & -1 \\ 0 & -3 & -6 & 4 & 9 \end{pmatrix}$$

$$\xrightarrow[R3 \mapsto R3+(2)R1]{R2 \mapsto R2+(1)R1} \begin{pmatrix} 1 & 4 & 5 & -9 & -7 \\ 0 & 2 & 4 & -6 & -6 \\ 0 & 5 & 10 & -15 & -15 \\ 0 & -3 & -6 & 4 & 9 \end{pmatrix}$$

$$\xrightarrow{R2 \mapsto \left(\frac{1}{2}\right) R2} \begin{pmatrix} 1 & 4 & 5 & -9 & -7 \\ 0 & 1 & 2 & -3 & -3 \\ 0 & 5 & 10 & -15 & -15 \\ 0 & -3 & -6 & 4 & 9 \end{pmatrix}$$

$$\xrightarrow[R4 \mapsto R4+(3)R2]{R3 \mapsto R3+(-5)R2} \begin{pmatrix} 1 & 4 & 5 & -9 & -7 \\ 0 & 1 & 2 & -3 & -3 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -5 & 0 \end{pmatrix}$$

$$\xrightarrow{R3 \leftrightarrow R4} \begin{pmatrix} 1 & 4 & 5 & -9 & -7 \\ 0 & 1 & 2 & -3 & -3 \\ 0 & 0 & 0 & -5 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$

3 pivots

pivot columns are C1, C2, C4.

## Important Terms

pivot position (or simply pivot): a position of a leading entry in an echelon form pivot column: a column that contains a pivot.

Basic variable: a variable that corresponds to a pivot column.

Free variable: a non-basic variable.

Example

$$\begin{array}{c}\left(\begin{array}{cccc|c} 1 & 4 & 5 & -9 & -7 \\ 0 & 1 & 2 & -3 & -3 \\ 0 & 0 & 0 & -5 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right) \\ \begin{array}{ccccc} x_1 & x_2 & x_3 & x_4 & \phantom{0} \end{array}\end{array}$$

basic variables: $x_1, x_2, x_4$

free variable: $x_3$.

$\boxed{\text{Fact}}$

- A linear system has no solution (inconsist.)
  $\Longleftrightarrow$ echelon form of the augmented matrix contains a row like
  $$(0 \ 0 \ 0 \ \cdots \ 0 \ c), \quad c \neq 0$$

- If the system is consistent, then

  a) it has infinitely many solutions
  $\Longleftrightarrow$ there is a free variable

  b) it has exactly one solution
  $\Longleftrightarrow$ there is no free variable.

Example

$$\begin{pmatrix} 1 & 1 & 2 \\ 0 & 0 & 0 \end{pmatrix}$$ augmented matrix

- consistent.
- $x_1$ is basic. $x_2$ is free

$\Longrightarrow$ infinitely many solutions

$$\begin{cases} x_1 + x_2 = 2 \\ \phantom{x} \end{cases} \iff x_1 = 2 - x_2$$

Infinitely many solutions.

$(1, 1)$
$(0, 2)$
$(-1, 3)$
$\vdots$

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<u>Example</u> Reduce the matrix to REF.

$$\begin{pmatrix} 0 & 3 & -6 & 6 & 4 & -5 \\ 3 & -7 & 8 & -5 & 8 & 9 \\ 3 & -9 & 12 & -9 & 6 & 15 \end{pmatrix}$$

<u>Sol.</u>

$$\longrightarrow \begin{pmatrix} 3 & -9 & 12 & -9 & 6 & 15 \\ 3 & -7 & 8 & -5 & 8 & 9 \\ 0 & 3 & -6 & 6 & 4 & -5 \end{pmatrix}$$

$$\longrightarrow \begin{pmatrix} 3 & -9 & 12 & -9 & 6 & 15 \\ 0 & 2 & -4 & 4 & 2 & -6 \\ 0 & 3 & -6 & 6 & 4 & -5 \end{pmatrix}$$

$$\longrightarrow \begin{pmatrix} 3 & -9 & 12 & -9 & 6 & 15 \\ 0 & 1 & -2 & 2 & 1 & -3 \\ 0 & 3 & -6 & 6 & 4 & -5 \end{pmatrix}$$

$$\begin{pmatrix} 3 & -9 & 12 & -9 & 6 & 15 \end{pmatrix}$$

$$\longrightarrow \begin{pmatrix} 3 & -9 & 12 & -9 & 6 & 15 \\ 0 & 1 & -2 & 2 & 1 & -3 \\ 0 & 0 & 0 & 0 & 1 & 4 \end{pmatrix}$$ echelon form.

$$\xrightarrow[R1 \mapsto R1+(-6)R3]{R2 \mapsto R2+(-1)R3} \begin{pmatrix} 3 & -9 & 12 & -9 & 0 & -9 \\ 0 & 1 & -2 & 2 & 0 & -7 \\ 0 & 0 & 0 & 0 & 1 & 4 \end{pmatrix}$$

$$\xrightarrow{R1 \mapsto R1+(9)R2} \begin{pmatrix} 3 & 0 & -6 & 9 & 0 & -72 \\ 0 & 1 & -2 & 2 & 0 & -7 \\ 0 & 0 & 0 & 0 & 1 & 4 \end{pmatrix}$$

$$\xrightarrow{R1 \mapsto \left(\frac{1}{3}\right) R1} \begin{pmatrix} 1 & 0 & -2 & 3 & 0 & -24 \\ 0 & 1 & -2 & 2 & 0 & -7 \\ 0 & 0 & 0 & 0 & 1 & 4 \end{pmatrix}$$ REF

## Solutions of Linear System

*Example*

$$\begin{pmatrix} 1 & 6 & 0 & 3 & 0 & 0 \\ 0 & 0 & 1 & -8 & 0 & 5 \\ 0 & 0 & 0 & 0 & 1 & 7 \end{pmatrix}$$

basic variables: $x_1, x_3, x_5$

free variables: $x_2, x_4$

$$\begin{cases} x_1 + 6x_2 + 3x_4 = 0 & \Leftrightarrow x_1 = -6x_2 - 3x_4 \\ x_3 - 8x_4 = 5 & \Leftrightarrow x_3 = 5 + 8x_4 \\ x_5 = 7 \end{cases}$$

$$\Longrightarrow \begin{cases} x_1 = -6x_2 - 3x_4 \\ x_2 \text{ is free} \\ x_3 = 5 + 8x_4 \end{cases}$$

General Solution

$$\Rightarrow \begin{cases} x_2 \text{ is free} \\ x_3 = 5 + 8x_4 \\ x_4 \text{ is free} \\ x_5 = 7 \end{cases}$$

General Solution

e.g. choose $x_2 = 1$, $x_4 = 0$

Then $x_1 = -6$, $x_3 = 5$, $x_5 = 7$.

$(-6, 1, 5, 0, 7)$ is a solution.

Example

$$\begin{cases} 3x_1 + 4x_2 = -3 \\ 2x_1 + 5x_2 = 5 \\ -2x_1 - 3x_2 = 1 \end{cases}$$

$$\begin{pmatrix} 3 & 4 & -3 \\ 2 & 5 & 5 \\ -2 & -3 & 1 \end{pmatrix} \rightsquigarrow \begin{pmatrix} 3 & 4 & -3 \\ 0 & 1 & 3 \\ 0 & 0 & 0 \end{pmatrix}$$

echelon form.

Consistent. no free variable

$\Rightarrow$ exactly one solution

$$\rightsquigarrow \begin{pmatrix} 3 & 0 & -15 \\ 0 & 1 & 3 \\ 0 & 0 & 0 \end{pmatrix} \longrightarrow \begin{pmatrix} 1 & 0 & -5 \\ 0 & 1 & 3 \\ 0 & 0 & 0 \end{pmatrix}$$

REF.

$$\begin{cases} x_1 = -5 \\ x_2 = 3 \end{cases}$$

the solution.

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## Examples

a) How many pivots can a $4 \times 6$ matrix have? At most 4

b) " " " $6 \times 4$ matrix have? At most 4.

c) A system has 3 equations and 4 variables. How many solutions does it have if it is consistent?

$$\begin{pmatrix} * & * & * & * & * \\ * & * & * & * & * \\ * & * & * & * & * \end{pmatrix}$$

augmented matrix

Infinitely many solutions.