MATH 1005A
Test 1 Solutions

February 2, 2017

[Marks]

[6] 1. Find the general solution of the equation $y' = 2xy$, and its orthogonal trajectories.

Solution:

$y' = 2xy \;\Rightarrow\; \dfrac{y'}{y} = 2x \;\Rightarrow\; \ln|y| = x^2 + c \;\Rightarrow\; y = ke^{x^2}$. The orthogonal trajectories satisfy $y' = -\dfrac{1}{2xy} \;\Rightarrow\; 2yy' = -\dfrac{1}{x} \;\Rightarrow\; y^2 = -\ln|x| + c$, or $y = \pm\sqrt{c - \ln|x|}$.

[6] 2. Solve the initial-value problem $y' = \dfrac{3x^2 + y^2}{xy}$, $y(1) = 3$.

Solution:

$y' = \dfrac{3x^2 + y^2}{xy} = 3\dfrac{x}{y} + \dfrac{y}{x}$, $u = \dfrac{y}{x} \Rightarrow u + xu' = \dfrac{3}{u} + u \Rightarrow uu' = \dfrac{3}{x} \Rightarrow \dfrac{1}{2}u^2 = 3\ln|x| + c$ $\Rightarrow u = \pm\sqrt{6\ln|x| + k} \Rightarrow y = \pm x\sqrt{6\ln|x| + k}$. $y(1) = 3 \Rightarrow k = 9$ and $y = x\sqrt{6\ln|x| + 9}$.

[6] 3. Find the general solution of the equation $x^4y' + 2x^3y = 1$, $x > 0$.

Solution:

The standard form is $y' + \dfrac{2}{x}y = \dfrac{1}{x^4}$, with the integrating factor $e^{\int \frac{2}{x}\,dx} = e^{2\ln(x)} = x^2$, and the equation becomes $x^2y' + 2xy = \dfrac{1}{x^2}$, i.e., $(x^2y)' = \dfrac{1}{x^2} \;\Rightarrow\; x^2y = -\dfrac{1}{x} + c \;\Rightarrow$ $y = -\dfrac{1}{x^3} + \dfrac{c}{x^2}$.

[6] 4. Find the general solution of the equation $3x^2y + y^2 + 1 + (x^3 + 2xy + 3y^2)y' = 0$.

Solution:

$P_y = 3x^2 + 2y = Q_x \;\Rightarrow\;$ the equation is exact. Then $f_x = P = 3x^2y + y^2 + 1 \;\Rightarrow$ $f(x, y) = x^3y + xy^2 + x + g(y)$, and $f_y = Q \;\Rightarrow\; x^3 + 2xy + g'(y) = x^3 + 2xy + 3y^2 \;\Rightarrow$ $g(y) = y^3 + c_1 \;\Rightarrow\; f(x, y) = x^3y + xy^2 + x + y^3 + c_1$, and the general solution of the equation is $x^3y + xy^2 + x + y^3 = c$.

[6] 5. Find an integrating factor which makes the equation $3x^2e^y + x + e^yy' = 0$ exact. Do not solve the equation.

Solution:

$\dfrac{P_y - Q_x}{Q} = \dfrac{3x^2e^y}{e^y} = 3x^2 \;\Rightarrow\;$ an integrating factor $I(x)$ exists and is determined by $\dfrac{I'(x)}{I(x)} = 3x^2 \;\Rightarrow\; \ln|I(x)| = x^3 \;\Rightarrow\; I(x) = e^{x^3}$.