CONCORDIA UNIVERSITY
Department of Economics

Solution

ECON 222 SECTIONS C, D
STATISTICAL METHODS II
Winter 2019 – MIDTERM 1
Sunday, February 17th, 9:30am – 11:30am

Student Name: **Student ID:** **Section:**

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1. **(8 marks)** Differentiate the following functions with respect to x.

a. **(2 marks)** $f(x) = 1/(x^3)$

$$f'(x) = -3\,(x^{-4}) = \frac{-3}{x^4}$$

b. **(2 marks)** $f(x) = x \ln (x^4 + 3x^2)$

$$f'(x) = \ln(x^4+3x^2) + x\left[\frac{4x^3+6x}{x^4+3x^2}\right]$$

c. **(2 marks)** $f(x) = e^{5x^3}$

$$f'(x) = 15x^2 e^{5x^3}$$

d. **(2 marks)** $f(x) = e^x/x^2$

$$f'(x) = \frac{x^2e^x - 2xe^x}{x^4} = \frac{xe^x(x-2)}{x^4}$$

$$\Rightarrow f'(x) = \frac{e^x(x-2)}{x^3}$$

2. **(4 marks)** Solve the following:

a. **(2 marks)** $\int_1^2 2x^3 + \frac{1}{x} dx$

$$= 2\int_1^2 x^3 dx + \int_1^2 \frac{1}{x} dx$$

$$= 2\left[\frac{x^4}{4}\right]_1^2 + \left[\ln x\right]_1^2$$

$$= 2\left[4 - \frac{1}{4}\right] + 0.693 = 8.193$$

b. **(2 marks)** $\int_1^5 \frac{15x^2 + 20x}{x^3 + 2x^2} dx$

$$= \int_1^5 \frac{5(3x^2 + 4x)}{x^3 + 2x^2} dv$$

$$= 5\int_1^5 \frac{3x^2 + 4x}{x^3 + 2x^2} dx = 5 \ln\left[x^3 + 2x^2\right]_1^5$$

$$= 5\left[\ln(175) - \ln(3)\right] = 20.331$$

3. **(6 marks)** Let X be a continuous random variable with a probability density function f(x) = $4x^3/255$ where $1 \leq x \leq 4$.

a. **(2 marks)** Compute P $(2 \leq x \leq 3)$

$$\frac{4}{255}\int_2^3 x^3 dx = \frac{4}{255}\left[\frac{x^4}{4}\right]_2^3 = \frac{4}{255}\left[16.25\right] = 0.2549$$

b. **(2 marks)** Find the cumulative distribution function of x, F(x).

$$F(x) = \frac{4}{255}\int_1^x x^3 dx = \frac{4}{255}\left[\frac{x^4}{4}\right]_1^x$$

$$\Rightarrow F(x) = \frac{4}{255}\left[\frac{x^4}{4} - \frac{1}{4}\right] = \frac{x^4 - 1}{255}$$

c. **(2 marks)** Find the probability density function of Y where Y = 1/x.

I find $w(y)$: $x = \frac{1}{y}$

II find $\frac{dw(y)}{dy} = \frac{dx}{dy} = \frac{-1}{y^2}$

III Construct $h(y) = f(w(y)) \left| \frac{dw(y)}{dy} \right|$

$\Rightarrow h(y) = \frac{4\left(\frac{1}{y}\right)^3}{255} \left| \frac{-1}{y^2} \right| = \frac{4}{255\, y^5}$. $\quad \frac{1}{4} \leq Y \leq 1$

4. **(9 marks)** Suppose that X and Y are two random variables with joint density $f(x,y) = 6x^2y$ where $0 \leq x \leq 1$ and $0 \leq y \leq 1$,

a. **(3 marks)** Briefly explain whether X and Y are independent.

$f_x(x) = 6x^2 \int_0^1 y\, dy = 6x^2 \left[\frac{y^2}{2}\right]_0^1 = 3x^2$

$f_y(y) = 6y \int_0^1 x^2\, dx = 6y \left[\frac{x^3}{3}\right]_0^1 = 2y$

Note that $f_x(x) f_y(y) = (3x^2)(2y) = 6x^2y = f(x,y)$

$\Rightarrow$ X & Y are independent.

b. **(3 marks)** Find the conditional mean and the conditional variance of X given Y=1/2.

$f(x|y) = \frac{f(x,y)}{f_y(y)} = \frac{6x^2y}{2y} = 3x^2$

Note that it does not depend on y which is expected given your answer in (a).

$E(x|y=\frac{1}{2}) = \int_0^1 x(3x^2)\, dx = 3\left[\frac{x^4}{4}\right]_0^1 = \frac{3}{4}$

$E(x^2|y=\frac{1}{2}) = \int_0^1 (x^2)(3x^2)\, dx = 3\left[\frac{x^5}{5}\right]_0^1 = \frac{3}{5}$

$Var(x|y=\frac{1}{2}) = \frac{3}{5} - \left(\frac{3}{4}\right)^2 = 0.0375$

c. **(3 marks)** Calculate the P (Y > 0.5 | X = 0.75).

$$P(y \mid x) = \frac{6x^2y}{3x^2} = 2y$$

$$\Rightarrow P(Y > 0.5 \mid X = 0.75) = 1 - P(0 \leq Y \leq 0.5 \mid X = 0.75)$$

$$= 1 - \int_0^{0.5} 2y\,dy = 1 - \left[y^2\right]_0^{0.5}$$

$$= 1 - 0.25 = 0.75$$

5. **(6 marks)** Suppose that X_1, X_2, X_3, X_4 is a random sample from a normally distributed population with mean (μ) and variance (σ2). The observations in the sample are independent and are sorted in ascending order where X_1 is the smallest observation and X_4 is the largest. compare between the following estimators, which one will you choose? Explain.

   i. $\tilde{X} = (1/2)\,X_1 + (1/2)\,X_4$,
   ii. $\hat{X} = (1/4)\,X_1 + (1/4)\,X_2 + (1/4)\,X_3 + (1/4)\,X_4$.

$$E(\tilde{X}) = \frac{1}{2}E(X_1) + \frac{1}{2}E(X_2) = \frac{1}{2}\mu + \frac{1}{2}\mu = \mu$$

$$E(\hat{X}) = \frac{1}{4}E(X_1) + \frac{1}{4}E(X_2) + \frac{1}{4}E(X_3) + \frac{1}{4}E(X_4)$$

$$= \frac{1}{4}(4\mu) = \mu$$

$\Rightarrow \tilde{X}$ & $\hat{X}$ are unbiased.

$$Var(\tilde{X}) = \frac{1}{4}Var(X_1) + \frac{1}{4}Var(X_2) \quad \because X_1, X_2 \text{ are independen}$$

$$= \frac{1}{4}\left[2\sigma^2\right] = \frac{\sigma^2}{2}$$

$$Var(\hat{X}) = \frac{1}{16}\left[4\sigma^2\right] = \frac{\sigma^2}{4}$$

Since $Var(\hat{X}) < Var(\tilde{X}) \Rightarrow \hat{X}$ is more efficient.

6. **(14 marks)** The risk management department in a major bank wants to estimate the average monthly losses on the bank's mortgage portfolio. The monthly losses on the portfolio, in millions of dollars, over the past twelve months are given in the following table.:

Jan.	Feb.	Mar.	April	May	June	July	Aug.	Sept.	Oct.	Nov.	Dec.
10	2	5	8	7	1	15	2	3	4	9	11

a. **(2 marks)** Calculate the sample average.

X : Monthly losses

$$\bar{X} = \frac{10+2+5+8+7+1+15+2+3+4+9+11}{12} = \frac{77}{12} = 6.417$$

b. **(2 marks)** Calculate the sample variance.

$$s^2 = \frac{\sum (X-\bar{X})^2}{N-1} = \frac{204.9167}{11} = 18.62879$$

c. **(2 marks)** A senior analyst in the team assumes that losses are normally distributed and are independent. Estimate the population mean and the population variance.

$\bar{X} = 6.417$ is an unbiased estimator of μ

$s^2 = 18.62879$ is an unbiased estimator of σ^2

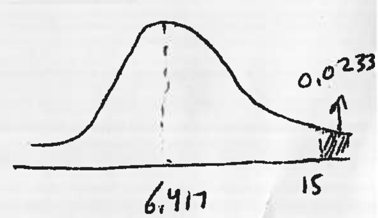
d. **(2 marks)** Compute the probability that losses would exceed 15 million dollars in a given month.

Note that we assume $X \sim N(\mu, \sigma^2) \Rightarrow$ We could use the unbiased estimates of μ & σ^2

$\Rightarrow X \sim N(6.417, 18.62879)$

$$\Rightarrow P(X > 15) = 1 - P(X \leq 15) = 1 - P\left(Z \leq \frac{15 - 6.417}{\sqrt{18.62879}}\right)$$

$$= 1 - P(Z \leq 1.9886) = 1 - 0.9767 = 0.0233$$

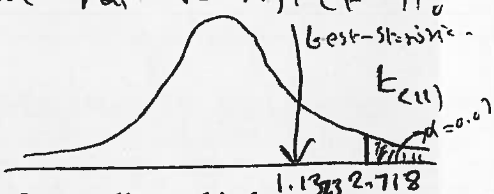


e. **(2 marks)** Test the following with 99% confidence, explain your choice of the constructed statistic and demonstrate your answer graphically.

$H_0 : \mu = 5$

$H_1 : \mu > 5$ $\Rightarrow$ Right tail test

- $\alpha = 1 - 0.99 = 0.01$
- test-statistic $= \dfrac{\bar{X} - 5}{se(\bar{X})} = \dfrac{6.417 - 5}{4.316/\sqrt{12}} = \dfrac{1.417}{1.248} = 1.1373 \sim t_{(11)}$
- $t_c^{(\alpha = 0.01)}$: $P(t > t_c^{\alpha}) = 0.01 \Rightarrow t_c^{\alpha} = 2.718$
- $\because$ test-statistic $< t_c^{\alpha} \Rightarrow$ we fail to reject H_0



f. **(4 marks)** Another senior analyst questioned the assumptions of normality and independence of the monthly losses. Discuss the consequences on your answers in parts (c, d, and e) in the case the losses are not normally distributed and are dependent.

If X does not follow a normal distribution and the draws are not independent:

- ~~inperfect~~ $\bar{X}$ & S^2 are still unbiased estimators of μ & σ^2 respectively. However $Var(\bar{X}) \neq \frac{S^2}{N}$ and $\bar{X}$ is not normally distributed. Also note that we can not use the Central limit theorem because the sample is small.
- Consequently, the probability of extreme losses that we obtained in part (d) is invalid.
- Similarly, the result in part (e) is also invalid because the used $se(\bar{X})$ is wrong.