MAT 1330 - Calculus for the Life Sciences I

Notes — By Eric Hua

Contents

Introduction

Main Contents:

- Derivatives: product and quotient rules, chain rule, derivative of exponential, logarithm and basic trigonometric functions, higher derivatives, curve sketching.
- Applications of the derivative to life sciences.
- Discrete dynamical systems: equilibrium points, stability, cobwebbing.
- Integrals: indefinite and definite integrals, fundamental theorem of calculus, antiderivatives, substitution, integration by parts.
- Applications of the integral to life sciences.

Prerequisite: One of MAT1339, Ontario 4U Calculus and Vectors (MCV4U) or an equivalent. The courses MAT1330, MAT1300, MAT1308, MAT1320 cannot be combined for credits.

Precalculus Review

1. Real numbers and intervals

Interval Notation	**Set Notation**
$[a, b]$	$\{x \in \mathbb{R} : a \leq x \leq b\}$
(a, b)	$\{x \in \mathbb{R} : a < x < b\}$
$[a, b)$	$\{x \in \mathbb{R} : a \leq x < b\}$
$(a, b]$	$\{x \in \mathbb{R} : a < x \leq b\}$
$(a, +\infty)$	$\{x \in \mathbb{R} : x > a\}$
$[a, +\infty)$	$\{x \in \mathbb{R} : x \geq a\}$
$(-\infty, b)$	$\{x \in \mathbb{R} : x < b\}$
$(-\infty, b]$	$\{x \in \mathbb{R} : x \leq b\}$
$(-\infty, +\infty)$	$\mathbb{R}$

2. Solving inequalities

Example 1 *Solve the inequality*

$$-2x - 3 \leq -13.$$

Solution: We have

$$-2x - 3 \leq -13 \Rightarrow -2x \leq -13 + 3 \Rightarrow -2x \leq -10.$$

The next step would be to divide both sides by -2. Since $-2 < 0$, the sense of the inequality is inverted, and so

$$-2x \leq -10 \Rightarrow x \geq \frac{-10}{-2} \Rightarrow x \geq 5.$$

Example 2 *Solve the inequality*

$$x^2 + 2x - 35 < 0.$$

Solution: Observe that $x^2 + 2x - 35 = (x - 5)(x + 7)$, which vanishes when $x = -7$ or when $x = 5$. Now we construct the table:

$x \in$	$(-\infty, -7)$	$(-7, 5)$	$(5, +\infty)$
$x + 7$	$-$	$+$	$+$
$x - 5$	$-$	$-$	$+$
$(x + 7)(x - 5)$	$+$	$-$	$+$

On the last row, the sign of the product $(x+7)(x-5)$ is determined by the sign of each of the factors $x+7$ and $x-5$.

From the sign diagram above we see that

$$\{x \in \mathbb{R} : x^2 + 2x - 35 < 0\} = (-7, 5).$$

Notice that we exclude both $x = -7$ and $x = 5$ in the set, as $(x+7)(x-5)$ vanishes there.

3. Absolute Values

Definition 1 *Let* $x \in \mathbb{R}$. *The* absolute value of x*—denoted by* $|x|$*—is defined by*

$$|x| = \begin{cases} -x & \text{if } x < 0, \\ x & \text{if } x \geq 0. \end{cases}$$

Example 3 *Let* $x > 10$. *Then* $|3 - |5 - x|| = |3 - (x-5)| = |8 - x|$.

- $|x| \leq t \iff -t \leq x \leq t.$
- $|x| \geq t \iff x \geq t \qquad \text{or} \qquad x \leq -t.$
- Triangle Inequality: Let a, b be real numbers. Then $|a + b| \leq |a| + |b|$.

Example 4 *Solve the inequality* $|2x - 1| \leq 1$.

Solution:

$$|2x-1| \leq 1 \iff -1 \leq 2x - 1 \leq 1 \iff 0 \leq 2x \leq 2 \iff 0 \leq x \leq 1 \iff x \in [0, 1].$$

The solution set is the interval $[0, 1]$.

4. Exponents and radicals

Properties of exponents:

- $x^0 = 1, \quad x \neq 0.$
- $x^{-n} = \frac{1}{x^n}, \quad x \neq 0.$
- $x^{1/n} = \sqrt[n]{x}$, $x^{m/n} = \sqrt[n]{x^m}$.
- $x^m x^n = x^{m+n}$, $x^m / x^n = x^{m-n}$.
- $(x^m)^n = x^{mn}$.

- $x^n y^n = (xy)^n$.

For Example,

$$\frac{x^{3/2}+5x^2}{x^{1/2}} = x(1+5x^{1/2}).$$

5. Factoring Polynomials

- $a^2 - b^2 = (a-b)(a+b)$.
- $a^3 - b^3 = (a-b)(a^2+ab+b^2)$ and $a^3+b^3 = (a+b)(a^2-ab+b^2)$.
- $x^n - y^n = (x-y)(x^{n-1}+x^{n-2}y+x^{n-3}y^2+\cdots+x^2y^{n-3}+xy^{n-2}+y^{n-1})$.
- $(a \pm b)^2 = a^2 \pm 2ab + b^2$.
- $(a+b)^3 = a^3+3a^2b+3ab^2+b^3$ and $(a-b)^3 = a^3-3a^2b+3ab^2-b^3$.

Example 5

$$\begin{aligned} x^4+x^2+1 &= x^4+2x^2+1-x^2 \\ &= (x^2+1)^2-x^2 \\ &= (x^2+1-x)(x^2+1+x). \end{aligned}$$

Example 6 $x^2-8x-9 = (x-9)(x+1)$.

6. Rationalizing denominator or numerator

- If the denominator is $\sqrt{a}$, then multiply both top and bottom by $\sqrt{a}$.
- If the denominator is $\sqrt{a} \pm \sqrt{b}$, then multiply both top and bottom by $\sqrt{a} \mp \sqrt{b}$.

Example 7

$$\frac{x}{\sqrt{x+4}-2} = \frac{x(\sqrt{x+4}+2)}{(\sqrt{x+4}-2)(\sqrt{x+4}+2)} = \frac{x(\sqrt{x+4}+2)}{x} = \sqrt{x+4}+2.$$

1.1–2.3 Functions and Models

Definition of a Function

Function: *A function $y = f(x)$ from a set D to a set R is a rule that assigns a unique element $f(x) \in R$ to each element $x \in D$. (x is called independent variable, y is called dependent variable).*

- Domain of the function $y = f(x)$: D=the set of all values of the independent variable x for which the function is defined.
- Range of the function: R=the set of all values taking on by the dependent variable y.

Example: $f(x) = \frac{x^2}{x^2-3x+2}$ is a function, $D = \{x : x \neq 1, 2\}$.
Example: $f(x) = \pm x^2$ is not a function.
Some special functions:

- Linear function: $y = f(x) = mx + b$.
- Increasing function $f(x)$: $f(x)$ increases as x increases.
- Decreasing function $f(x)$: $f(x)$ decreases as x increases.
- Piecewise defined functions: $f(x) = \begin{cases} 2x, & x \leq 0; \\ 3x, & x > 0. \end{cases}$
- Power function: $f(x) = kx^p$, where $k \neq 0$ and p are constants, e.g., $\sqrt{1 - x^2}$.
- Polynomials $P(x) = a_n x^n + a_{n-1} x^{n-1} + ... + a_1 x + a_0$, where n is a positive integer (which is called the degree of $P(x)$).
- Rational function: $f(x) = \frac{p(x)}{q(x)}$.
- Absolute value:

$$|x| = \begin{cases} -x & \text{if } x < 0, \\ x & \text{if } x \geq 0. \end{cases}$$

Exponential functions

We say that $f(x) = a^x$ is an exponential function with base a.

- Domain: $x \in \mathbb{R}$; Range: $y > 0$.
- Exponential growth: $a > 1$; Exponential decay: $0 < a < 1$.
- Natural exponential function is defined as: $y = f(x) = e^x$, where $e \doteq 2.71828....$
- Exponential model: $f(x) = ce^{\alpha x}$, $c \neq 0$, $\alpha \neq 0$. Here α is the exponential growth/decay rate.
- Graph: e.g., $y = 2^x + 5$, $y = 2^{-x} + 5$.

Laws of exponents:

$$a^{x+y} = a^x a^y, \quad a^{x-y} = a^x/a^y, \quad (a^x)^y = a^{xy}, \quad a^x b^x = (ab)^x.$$

Example 8 *Solve for x:* $3^{2x-3} = 9^{1-3x}$.

Solution: $3^{2x-3} = 3^{2(1-3x)} \Rightarrow 2x - 3 = 2(1-3x), \Rightarrow 8x = 5, x = 5/8$.

Example 9 *Solve for x:* $2^{2x+1} - 9(2^x) + 4 = 0$.

Solution:
Let $z = 2^x$, then $2z^2 - 9z + 4 = 0$. $(2z-1)(z-4) = 0$, $z = 1/2, 4$. $x = -1, 2$.

Example 10 *The relationship between the length (inch) of Muskie fish and the weight (pound) can be modeled by*

$$W = 0.000089L^{3.325}.$$

E.g., $18lb \leftrightarrow 40in$.

Applications on population growth/decay: Let $P(t)$ be the population after t years.

- Half-life (exponential decay): The time required for the quantity to be reduced to half. Let H be the half-life, then

$$P(t+H) = \frac{1}{2}P(t) \Rightarrow P(t) = P_0(\frac{1}{2})^{t/H}.$$

- Doubling-time (exponential growth): The time required for the quantity to be doubled. Let D be the doubling time, then

$$P(t+D) = 2P(t) \Rightarrow P(t) = P_0(2)^{t/D}.$$

Example 11 *A bacterial culture starts with 500 bacteria and doubles in size every hour.*

a) How many are there after t hours?

b) How many are there after 10 minutes?

Solution: a) Let $P(t)$ be the number after t hours. Then $P(0) = 500$, $P(t+1) = 2P(t)$. $D = 1$.

$$P(t) = (500)2^{t/1} = (500)2^t.$$

b) $P(10/60) = (500)2^{10/60} = (500)2^{1/6}$.

Logarithms

Inverse function: One-to-one function: $y = f(x)$ is 1-1 $\Leftrightarrow$ for each $y \in R$, there is only one $x \in D$. Horizontal line test can be used to check this.

Example 12 *$f(x) = x^2$ is not 1-1; $g(x) = x^2$, $x > 0$ is 1-1.*

Inverse function: $y = f(x) \to x = f^{-1}(y)$. We write it as $y = f^{-1}(x)$.

- The graph of f^{-1} and the graph of f are symmetric about the line $y = x$.
- Cancelation: $f(f^{-1}(y)) = y$.
- $f^{-1}(f(x)) = x$
- $D(f) = R(f^{-1})$, $R(f) = D(f^{-1})$.

Example 13 *let $f(x) = \frac{3x+2}{5x-4}$, find the inverse $f^{-1}(x)$.*

Strategy:
1) Write $y = \frac{3x+2}{5x-4}$;
2) Switch x and y: $x = \frac{3y+2}{5y-4}$;
3) Isolate y: $y = \frac{4x+2}{5x-3}$;

4) Answer: $y = f^{-1}(x) = \frac{4x+2}{5x-3}$.

$$y = a^x \quad \xrightarrow{inversefunction} \quad y = \log_a x,$$
$$y = e^x \quad \xrightarrow{inversefunction} \quad y = \log_e x = \ln x,$$
$$y = 10^x \quad \xrightarrow{inversefunction} \quad y = \log_{10} x = \log x.$$

Definition: $y = \log_a x$ is called logarithmic function with the base a. Domain $= \{x > 0\}$.
Laws: Let $B, C > 0$. Then

1. $\log_a(BC) = \log_a B + \log_a C$,
2. $\log_a(\frac{B}{C}) = \log_a B - \log_a C$,
3. $\log_a(B^n) = n \log_a B$,
4. $\log_a(a^x) = x$, $\log_a a = 1$,
5. $a^{\log_a B} = B$,
6. $\log_a 1 = 0$.
7. Change of base: $\log_a b = \frac{\log_c b}{\log_c a}$.

Proof. Let $x = \log_a b$. Then $a^x = b \Rightarrow \log_c a^x = \log_c b \Rightarrow x \log_c a = \log_c b$.

Example 14 *Convert* a^x *to base* e.

$$a^x = e^{x \ln a}.$$

Example 15 *Simplify* $\log_3 18 - \log_3 2$.

Example 16 *Solve for* x*:*
(a) $3^{2x-1} = 4$,
(b) $\ln[\ln(2x+1)] = 1$,
(c) $\log_3 x + \log_3(x-8) = 2$.

Solution:
(a). $x = \frac{1}{2}(\ln 4 / \ln 3 + 1)$.
(b). $x = \frac{1}{2}(e^e - 1)$.
(c) $x = 9$.

Example 17 *Sketch* $y = \ln(x+1) - 2$.

Example 18 *Predict the population in 2010, if*

Year	*Population*
2000	*10*
2003	*10.5*

Solution: Let $P(t)$ be the population after t years. $t = 0 \Leftrightarrow 2000$, $P(0) = P_0 = 10, P(3) = 10.5$.

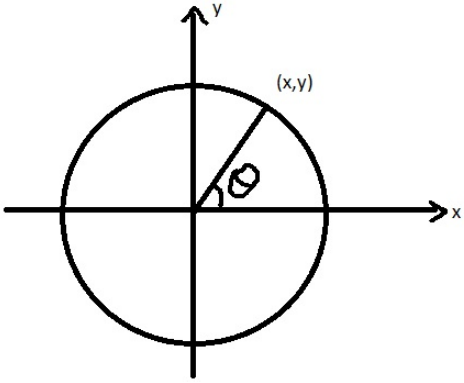
$$P(t) = P_0 a^t, \Rightarrow P(t) = 10a^t, \Rightarrow P(3) = 10a^3 = 10.5, \Rightarrow a \doteq 1.0164, \Rightarrow P(t) = 10(1.0164)^t.$$

$$P(10) = 10(1.0164)^{10} = 11.76648.$$

Trigonometric functions

Radian $\leftrightarrows$ Degree: t degree $= \frac{t}{180}\pi$.

For any point (x, y), let $r = \sqrt{x^2 + y^2}$.



$$\sin\theta = \frac{y}{r}, \quad \cos\theta = \frac{x}{r}, \quad \tan\theta = \frac{\sin\theta}{\cos\theta},$$
$$\sec\theta = \frac{1}{\cos\theta}, \quad \csc\theta = \frac{1}{\sin\theta}, \quad \cot\theta = \frac{1}{\tan\theta}.$$

Pythagorean trigonometric identity: $\sin^2 x + \cos^2 x = 1$.

Special values:

t	0	$\frac{\pi}{6}$	$\frac{\pi}{4}$	$\frac{\pi}{3}$	$\frac{\pi}{2}$
$\sin t$	0	$\frac{1}{2}$	$\frac{\sqrt{2}}{2}$	$\frac{\sqrt{3}}{2}$	1
$\cos t$	1	$\frac{\sqrt{3}}{2}$	$\frac{\sqrt{2}}{2}$	$\frac{1}{2}$	0

Addition formulas:

$$\sin(x+y) = \sin x \cos y + \cos x \sin y,$$

$$\cos(x+y) = \cos x \cos y - \sin x \sin y.$$

Double-angle formulas:

$$\sin 2x = 2 \sin x \cos x, \quad \cos 2x = \cos^2 x - \sin^2 x.$$

Half-angle formula.

$$\sin^2 x = \frac{1-\cos 2x}{2}, \quad \cos^2 x = \frac{1+\cos 2x}{2}.$$

Periods: $\sin x$ and $\cos x$ have period 2π, $\tan x$ and $\cot x$ have period π.

Example 19 *Find the period of the function* $f(x) = 2\sin[3(x - \frac{\pi}{6})] + 1$.

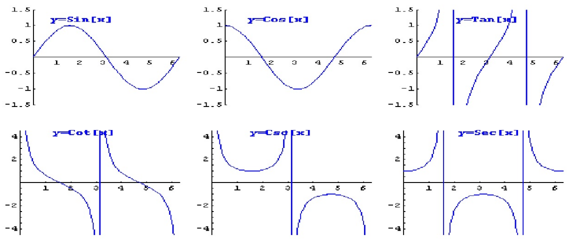
Example 20 *Find all values of* x *in the interval* $[0, 2\pi]$ *such that* $\sin^2 x - 3\cos^2 x = 0$.

Solution: $\frac{\pi}{3}, \frac{2\pi}{3}, \frac{4\pi}{3}, \frac{5\pi}{3}$.

Example 21 *Find* $\cos x$ *where* $x \in [\frac{\pi}{2}, 2\pi]$ *such that* $\sin x = 0.8$.

Solution: $\cos x = -0.6$

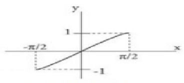
Graphs.



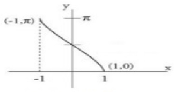
Inverse Trig Functions

Inverse Trig Function	Domain	Restriction (Range)	Meaning
$y = \arcsin x = \sin^{-1}(x)$	$-1 \leq x \leq 1$	$-\frac{\pi}{2} \leq y \leq \frac{\pi}{2}$	$\sin y = x$
$y = \arccos x = \cos^{-1}(x)$	$-1 \leq x \leq 1$	$0 \leq y \leq \pi$	$\cos y = x$
$y = \arctan x = \tan^{-1}(x)$	$-\infty < x < \infty$	$-\frac{\pi}{2} < y < \frac{\pi}{2}$	$\tan y = x$
$y = arcsec\ x = \sec^{-1}(x)$	$\|x\| \geq 1$	$0 \leq y \leq \pi, y \neq \frac{\pi}{2}$	$\sec y = x$
$y = arccsc\ x = \csc^{-1}(x)$	$\|x\| \geq 1$	$-\frac{\pi}{2} \leq y \leq \frac{\pi}{2},\ y \neq 0$	$\csc y = x$
$y = arccot\ x = \cot^{-1}(x)$	$-\infty < x < \infty$	$0 < y < \pi$	$\tan y = x$

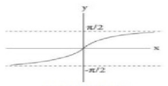
Graphs of the inverse functions: Using the symmetry line $y = x$ to get the graph for inverse from original functions.



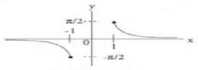
y = arcsin(x)



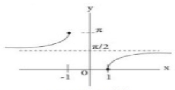
y = arccos(x)



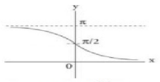
y = arctan(x)



y = arccsc(x)



y = arcsec(x)



y = arccot(x)

Example 22 *Find the exact values of the following expressions:*

(a) $\arcsin(1)$

(b) $\arctan(-1)$.

(c) $\tan^{-1}(\sqrt{3})$.

(d)$\sin[\cos^{-1}(\frac{\sqrt{3}}{2})]$.

(e) $\arctan(\tan x)$, *where* $\frac{3\pi}{4} \leq x \leq 2\pi$.

Example 23 *Simplify the following expression:* $\tan \arcsin \frac{x}{a}$.

Solution: Draw a right triangle with hypotenuse a and one side x. Let θ be the opposite angle of x. Then

$$\tan \arcsin \frac{x}{a} = \tan \theta = \frac{x}{\sqrt{a^2 - x^2}}.$$

3.1–3.5 Discrete-Time Dynamical System (DTDS)

Introduction

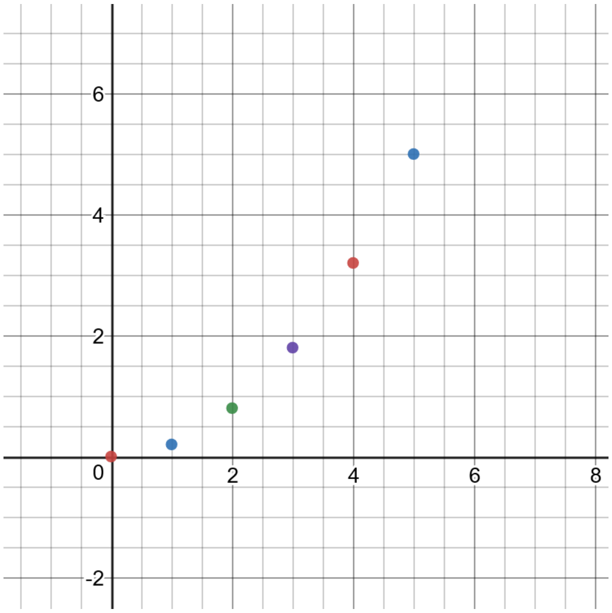
The dynamic of any situation refers to how the situation changes over the course of time. A dynamical system is a physical setting together with rules for how the setting changes or evolves from one moment of time to the next. One basic goal of the mathematical theory of dynamical systems is to determine or characterize the long-term behavior of the system. Often a physical setting is reduced to a set of measurements, for example, temperature, pressure, stock market prices, etc. In discrete-systems, we give these measurements at a sequence of specific times. We would hope that given the measurements at time t that we have a rule to determine the measurements at time $t+1$. If m_t represents the measurements at time t, this rule may take the form

$$m_{t+1} = f(m_t), \quad f^{-1}(m_{t+1}) = m_t,$$

where $f(x)$ is a given function fixed for all time, and is called **updating function**. This is referred as **recursion or recursive relation**. The **inverse** f^{-1} go one step into the past, which corrsponds to an "updating" function that goes backward in time.

Composition: $f \circ f$ = jump two time units into the future; $f \circ f \circ f$ = jump three time units into the future, ...

Solution and graph: The sequence m_0, m_1,... is the solution of the dynamical system. Graph $= \{(t, m_t) : t = 0, 1, 2, ...\}$.



Example 24 *Let* $f(x) = 2x(1-x)$. *We have some discrete systems like:*

$$x_0 = 0, x_1 = 0 = ... = x_n = ...;$$

$$x_0 = 1, x_1 = 0 = ... = x_n = ...;$$

$$x_0 = 0.5, x_1 = 0.5 = ... = x_n = ...$$

In general,

$$x_n = \underbrace{f \circ f \circ ... \circ f}_{n}(x_0).$$

Since $f(0) = 0$, $f(0.5) = 0.5$, so $x = 0$ and $x = 0.5$ are called fixed points of $f(x)$.

x_0	0.1
x_1	0.18
x_2	0.2952
x_3	0.41611392
x_4	0.4859262512
x_5	0.4996038592
x_6	0.4999996862
x_7	0.5000000000

We may easily guess the long-term behavior of this system:

$$\lim_{n\to\infty} x_n = 0.5.$$

Example 25 *Let* $x_{t+1} = 3x_t^2$, $x_0 = 0.2$. *Find* $f(x)$ *and* x_{100}.

Example 26 *Find the general solution:*

- *Basic exponential discrete-time dynamical system:* $b_{t+1} = rb_t$. *Solution:* $b_t = b_0 r^t$.
- *Basic additive discrete-time dynamical system:* $h_{t+1} = a + h_t$. *Solution:* $h_t = h_0 + at$.
- *Linear model:* $m_{t+1} = am_t + b$.

Solution:

$m_1 = am_0 + b,$

$m_2 = am_1 + b = a(am_0 + b) + b = a^2 m_0 + ab + b,$

$m_n = a^n m_0 + (a^{n-1} + ... + a + 1)b = a^n m_0 + \dfrac{(a^n - 1)b}{a - 1}.$

Example 27 *Dynamics of absorption of pain medication: Let* M_t *be the amount of methadone in the patient's body at time* t. *Due to absorption,* M_t *is reduced to half within a day. Administering a new dosage will increase that amount by 1. Then the model is*

$$M_{t+1} = 0.5M_t + 1.$$

Analysis of DTDS

Cobwebbing: A graphical solution technique

Given the discrete-time dynamical system

$$x_{t+1} = f(x_t)$$

and **initial condition** x_0, we want to find other points on the curve $y = f(x)$.

Strategy:

1. We draw the diagonal line $y = x$;

2. x_1 is the coordinate of the vertical point on the graph directly above x_0, so we get (x_0, x_1).

3. Move the point (x_0, x_1) horizontally until it intersects the diagonal line, we get the intersection (x_1, x_1).

4. Move the intersection vertically until it intersects the graph, we get x_2, then repeat...

5. **Sketch the solutions at times 0, 1, 2, ...**

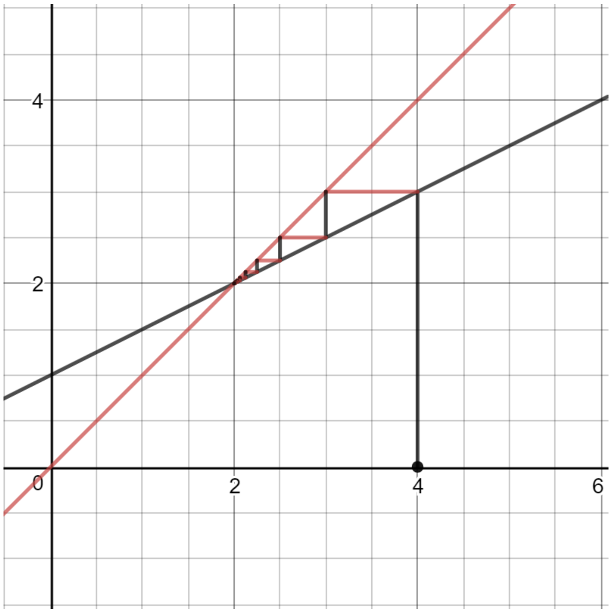
Example 28 *Cobweb the pain medication model with* $M_0 = 1$ *and* $M_0 = 4$.

Solution:

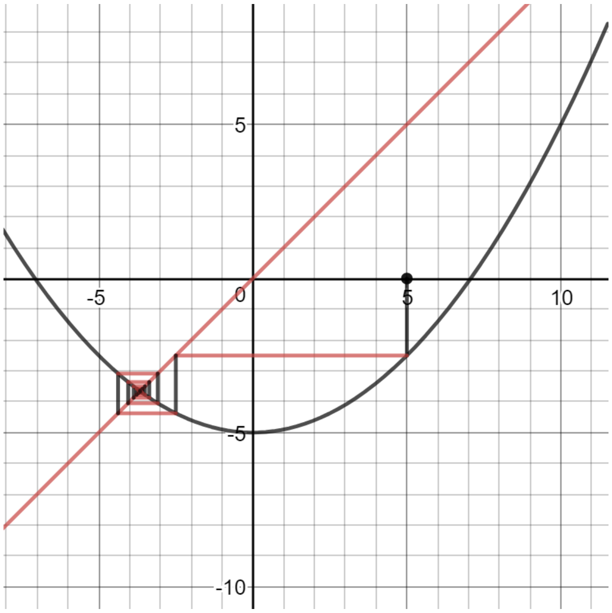
$$M_{t+1} = 0.5M_t + 1, \Rightarrow$$

$$1, 1.5, 1.75, 1.875, ...$$

$$4, 3, 2.25, 2.125, ...$$



Example 29 *Cobweb* $x_{t+1} = 0.1x_t^2 - 5$ *with* $x_0 = 5$.



Equilibrium (or fixed point):

Definition 2 *A point m^* is called an equilibrium (or fixed point) of the discrete-time dynamical system*

$$x_{t+1} = f(x_t)$$

if

$$f(m^*) = m^*.$$

Remark. At any equilibrium, $f(x)$ neither increases nor decreases, remains the same. The above definition gives you **Algebraic Approach** to find equilibria.

Graphic approach: The intersections of the updating function and the line $y = x$.

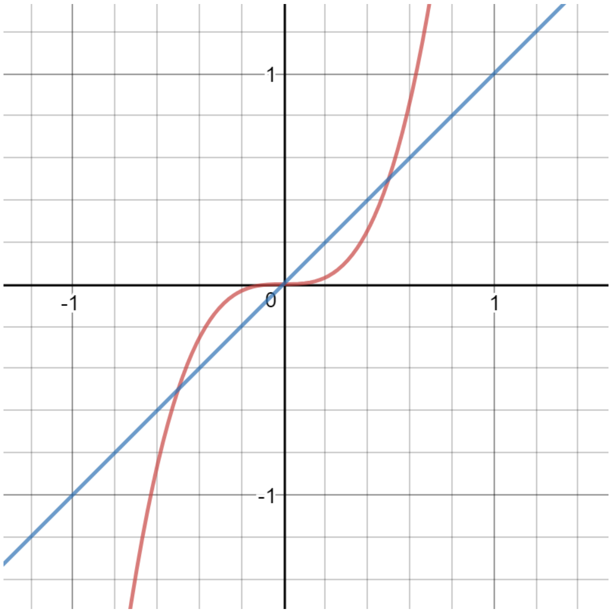
Example 30 $x_{t+1} = 4x_t^3$. *Find all equilibria.*

Solution: Step 1: Construct the equation $m^* = 4(m^*)^3$;

Step 2: Solve the equation, we obtain $0, 1/2, -1/2$.

Stability of equilibrium: An equilibrium m^* is called stable if the solutions that start near m^* stay near or approach m^*; if the solutions that start near m^* moves away from it, then m^* is unstable.

Example 31 $x_{t+1} = 4x_t^3$. *Study the stabilities of all equilibria.*



Solution: Stable at 0, unstable at $1/2, -1/2$.

Modeling with DTDS

Absorption of Caffeine: By c_t we denote the amount (in mg) of caffeine at time t (in hours). On average, our body eliminates 13% per hour. Assume that at the end of the same time interval we consume d extra mg of caffeine, then the model will be:

$$c_{t+1} = 0.87c_t + d.$$

Example 32 *Find the half life with* $d = 0$.

Solution: From the model $c_{t+1} = 0.87c_t$ we imply that

$$c_t = c_0(0.87^t).$$

$$\frac{1}{2} = 0.87^t, \Rightarrow t \approx 4.98$$

Population growth/decay: By b_t we denote the amount of bacterial at time t. Consider the model:

$$b_{t+1} = rb_t,$$

where r represents the number of new bacterial produced per bacterium, called the **per capita production**.

Example 33 *If the population doubles each hour, then* $r = 2$*; if the population decreases by 50% each hour, then* $r = 1/2$.

Alcohol Use: We define a unit of alcohol as: **one drink** contains 14 g of alcohol, which is equivalent to 44 mL of rum, or 144 mL of white wine, or 355 mL of beer. Let a_t be the amount of alcohol (in grams) at time t, let $r(a_t)$ be the rate of elimination when the amount of alcohol in the body is a_t. Then

$$r(a_t) = \frac{10.1}{4.2 + a_t}, \quad a_t \geq 5.9g.$$

Then

$$a_{t+1} = a_t - a_t r(a_t) + d(new\ amount) = a_t - \frac{10.1a_t}{4.2 + a_t} + d.$$

Example 34 *Assume that someone has two rapid drinks and then decides to consume half a drink every hour. What will the long-term effects be?*

Solution: $a_0 = 2(14) = 28$, $d = \frac{1}{2}(14) = 7$. Then

$$a_1 \approx 26.2174$$

$$a_2 \approx 24.5120$$

$$a_3 \approx 22.8894$$

The equilibrium is

$$a^* \approx 9.5$$

Nonlinear Dynamics Model of Selection

Discrete-time dynamical system is

- **liner**, if the updating function is linear;
- **nonliner**, if the updating function is nonlinear.

A model of selection: Let b_t and m_t be the population of bacterial and mutant respectively, at time t. Assume that

- **bacterial:** $b_{t+1} = rb_t$;
- **mutants:** $m_{t+1} = sm_t$.

If $s > r$, over time, the population of mutants will be larger and larger. The establishment of this mutant is an example of **selection**.

Modeling the dynamics of the fraction: Let p_t be the fraction of mutants at time t. Then

$$p_t = \frac{m_t}{m_t + b_t},$$

$$p_{t+1} = \frac{m_{t+1}}{m_{t+1} + b_{t+1}} = \frac{sp_t}{sp_t + r(1 - p_t)}.$$

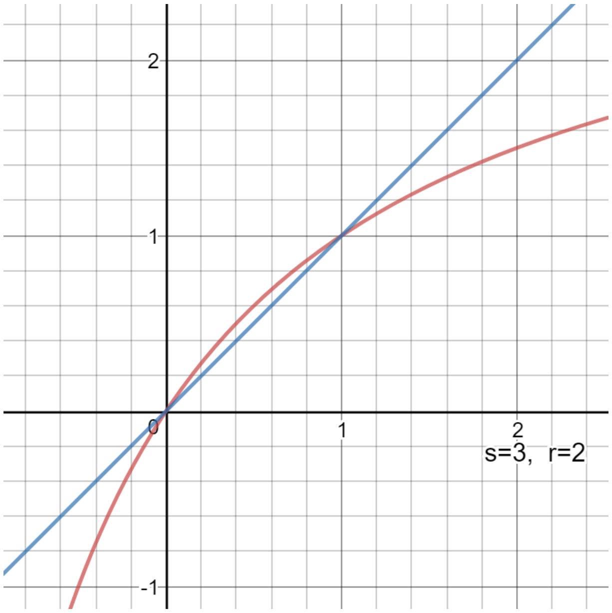
The updating function is

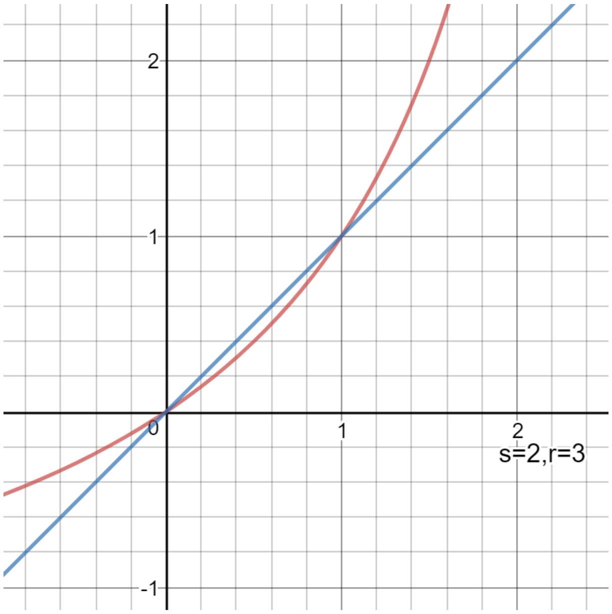
$$f(p_t) = \frac{sp_t}{sp_t + r(1 - p_t)}.$$

Equilibria are $p^* = 0, 1$(when $s \neq r$).

Stability of the equilibria:

- $p^* = 0$ **is unstable:** if $p_0 = 0.1$, then $(t, p_t) = (0, 0.1), ..., (\infty, 1)$;
- $p^* = 1$ **is stable:** if $p_0 = 0.8$, then $(t, p_t) = (0, 0.8), ..., (\infty, 1)$.


2
1
-1
0
1
2
s=3, r=2


2
1
-1
0
1
2
s=2,r=3

4.1–4.3 Limits

The Tangent and Velocity Problem

The average rate of change of $y = f(x)$ with respect to x over the interval $[x_1, x_2]$ is

$$\frac{\Delta y}{\Delta x} = \frac{y_2 - y_1}{x_2 - x_1} = \frac{f(x_2) - f(x_1)}{x_2 - x_1} = \frac{f(x_1 + h) - f(x_1)}{h}, h \neq 0.$$

Geometrically, it is the slope of the secant through two points $P(x_1, y_1)$ and $Q(x_2, y_2)$.

Instantaneous rates of change and tangent lines: What is a tangent line at point P on a curve? We chose another point Q on the curve. The line PQ is called a secant line. When Q tends to P, the secant PQ will tends to a line, which is called a the tangent line of the curve at P.

Definition 3 *Let $s = f(t)$ be position function.*

$$\text{average velocity} = \frac{\text{total distance}}{\text{total time}} = \frac{\Delta s}{\Delta t}.$$

Example 35 *Consider the position function $s = t^2 - 3t + 5$. Find the average velocity from $t = 3$ to $t = 4$.*

Solution: $\bar{v} = \frac{\Delta s}{\Delta t} = \frac{s(4) - s(3)}{4-3}$.

The Limit of A Function

Definition 4 *We write*

$$f(a - 0) = \lim_{x \to a^-} f(x) = L$$

and say that the limit of $f(x)$ is L as x approaches a from the left. Similarly, We write

$$f(a + 0) = \lim_{x \to a^+} f(x) = L$$

and say that the limit of $f(x)$ is L as x approaches a from the right.

Example 36 *Consider the Heaviside function*

$$H(t) = \begin{cases} 0, & t < 0; \\ 1, & t \geq 0. \end{cases}$$

$$\lim_{t \to 2^-} H(t) = 1,$$

$$\lim_{t \to 0^+} H(t) = 1, \lim_{t \to 0^-} H(t) = 0.$$

Example 37 $\lim_{x \to 0^-} \frac{|x|}{x} = -1, \lim_{x \to 0^+} \frac{|x|}{x} = 1.$

Example 38 *Let*

$$f(x) = \begin{cases} x - 5, & x < 0; \\ x^2 + 3x, & 0 \leq x \leq 1; \\ x^4 - x^3 + 4, & x > 1. \end{cases}$$

Then $\lim_{x \to 0^-} f(x) = -5$ *and* $\lim_{x \to 1^+} f(x) = 4$.

Definition 5 *We write*

$$\lim_{x \to a} f(x) = L$$

and say "as x *approaches* a*, the limit of* $f(x)$ *is* L*." If* L *is a finite number, we say that the limit exists, otherwise, the limit does not exist.*

Theorem 1

$$\lim_{x \to a} f(x) = L \Leftrightarrow \lim_{x \to a^-} f(x) = L \text{ and } \lim_{x \to a^+} f(x) = L.$$

Example 39 *Consider the Heaviside function*

$$H(t) = \begin{cases} 0, & t < 0; \\ 1, & t \geq 0. \end{cases}$$

$$\lim_{t \to 2} H(t) = 1,$$

$$\lim_{t \to 0^+} H(t) = 1, \lim_{t \to 0^-} H(t) = 0, \Rightarrow \lim_{t \to 0} H(t) \nexists.$$

Example 40 $\lim_{x \to 0} \frac{|x|}{x} \nexists.$

$$\because \lim_{x \to 0^-} \frac{|x|}{x} = -1, \lim_{x \to 0^+} \frac{|x|}{x} = 1.$$

Example 41 *Let*

$$f(x) = \begin{cases} x-5, & x<0; \\ x^2+3x, & 0 \le x \le 1; \\ x^4 - x^3 + 4, & x > 1. \end{cases}$$

Then $\lim_{x \to 0} f(x) \nexists$ *and* $\lim_{x \to 1} f(x) = 4$.

Solution: $\lim_{x \to 0^-} f(x) = \lim_{x \to 0^-} x - 5 = -5$ and $\lim_{x \to 0^+} f(x) = \lim_{x \to 0^-} x^2 + 3x = 0$. Similarly, $\lim_{x \to 1^-} f(x) = 4$ and $\lim_{x \to 1^+} f(x) = 4$.

Euler's Number e

$$e = \lim_{x \to 0} (1+x)^{1/x} = \lim_{x \to \infty} (1 + \frac{1}{x})^x = 2.71828....$$

Example 42 *Calculate*

$$\lim_{x \to 0} (1-x)^{1/x}, \quad \lim_{x \to \infty} (1 + \frac{3}{x})^x.$$

Example 43 *Evaluate*

$$\lim_{x \to 0^-} e^{1/x}.$$

Solution: Let $t = 1/x$, then $x \to 0^- \Leftrightarrow t \to -\infty$.

$$\lim_{x \to 0^-} e^{1/x} = \lim_{t \to -\infty} e^t = 0.$$

Limit Laws: Suppose that $\lim_{x \to a} f(x) \ \exists$ and $\lim_{x \to a} g(x) \ \exists$.

- $\lim_{x \to a} P(x) = P(a), \quad P(x)$ is a polynomial.
- $\lim_{x \to a} (cf(x) \pm dg(x)) = c \lim_{x \to a} f(x) \pm d \lim_{x \to a} g(x)$, c, d are constants.
- $\lim_{x \to a} [f(x)g(x)] = \lim_{x \to a} f(x) \cdot \lim_{x \to a} g(x)$.
- $\lim_{x \to a} \frac{f(x)}{g(x)} = \frac{\lim_{x \to a} f(x)}{\lim_{x \to a} g(x)}, \quad if \ \lim_{x \to a} g(x) \neq 0$.
- $\lim_{x \to a} [f(x)]^n = [\lim_{x \to a} f(x)]^n$.
- $\lim_{x \to a} \sqrt[n]{f(x)} = \sqrt[n]{\lim_{x \to a} f(x)}$. When n is even, we assume that $\lim_{x \to a} f(x) \neq 0$.

Example 44

$$\lim_{x\to 1}(x^2-3) = 1^2-3 = -2, \quad \lim_{x\to 1}\frac{3x^4+8x-2}{x-2} = \frac{3(1)^4+8(1)-2}{1-2} = -9.$$

Example 45 *Assume that* $\lim_{x\to a} f(x)$ *and* $\lim_{x\to a} g(x)$ *exist. If* $\lim_{x\to a}(f(x)+g(x)) = 3$, $\lim_{x\to a}(f(x)-g(x)) = 4$, *find* $\lim_{x\to a} f(x)g(x)$.

Solution: Note that

$$f(x) = \frac{1}{2}(f(x)+g(x)) + \frac{1}{2}(f(x)-g(x)), \quad f(x) = \frac{1}{2}(f(x)+g(x)) - \frac{1}{2}(f(x)-g(x)),$$

we imply that

$$\lim_{x\to a} f(x) = \frac{7}{2}, \quad \lim_{x\to a} g(x) = -\frac{1}{2}.$$

Thus

$$\lim_{x\to a} f(x)g(x) = -\frac{7}{4}.$$

Special case:

$$\lim_{x\to a}\frac{f(x)}{g(x)} \quad \textit{where } g(a) = 0.$$

- If $f(a) \neq 0$, then $\lim_{x\to a}\frac{f(x)}{g(x)}$ does not exist.
- If $f(a) = 0$, then simplify $\frac{f(x)}{g(x)}$ first, then study the limit.

Example 46

$$\lim_{x\to 2}\frac{3x^4+8x-2}{x-2} \nexists, \quad \lim_{x\to 2}\frac{x-2}{x-2} = 1.$$

Example 47 *Calculate the following limits:*

$$\lim_{x\to 2}\frac{x^2-4}{3-|x-5|}$$

$$\lim_{h\to 0}\frac{(h+1)^2-1}{h}$$

$$\lim_{x\to 0}\frac{\sqrt{x+4}-2}{x}$$

Solution:

$$\begin{aligned}
\lim_{x\to 2}\frac{x^2-4}{3-|x-5|}=\lim_{x\to 2}\frac{x^2-4}{x-2} &= \lim_{x\to 2}(x+2)=4,\\
\lim_{h\to 0}\frac{(h+1)^2-1}{h} &= \lim_{h\to 0}\frac{h(h+2)}{h}=\lim_{h\to 0}(h+2)=2,\\
\lim_{x\to 0}\frac{\sqrt{x+4}-2}{x} &= \lim_{x\to 0}\frac{(\sqrt{x+4}-2)(\sqrt{x+4}+2)}{x(\sqrt{x+4}+2)}=\lim_{x\to 0}\frac{x}{x(\sqrt{x+4}+2)}\\
&= \lim_{x\to 0}\frac{1}{\sqrt{x+4}+2}=\frac{1}{4}.
\end{aligned}$$

Theorem 2 *If $f(x)\le g(x)$ near $x=a$, then*

$$\lim_{x\to a}f(x)\le\lim_{x\to a}g(x).$$

Theorem 3 *The Sandwich Theorem (The Squeeze Theorem): If $f(x)\le g(x)\le h(x)$ near $x=a$, and $\lim_{x\to a}f(x)=\lim_{x\to a}h(x)=L$, then $\lim_{x\to a}g(x)=L$.*

Example 48 *Show that*

$$\lim_{x\to 0}x^4\cos\frac{3}{x}=0$$

by the Squeeze Theorem.

Solution: $-x^4\le x^4\cos\frac{3}{x}\le x^4$.

Example 49

$$\lim_{x\to 0}\sin x=0,\quad \lim_{x\to 0}\cos x=1.$$

Example 50 *Estimate the limit of*

$$\lim_{x\to 0}\frac{\sin x}{x}.$$

x	$\frac{\sin x}{x}$
1	0.84147098
0.1	0.99833417
0.01	0.99998333
0.001	0.99999983

Basic result:

$$\lim_{h\to 0}\frac{\sin h}{h}=1.$$

Proof. It is from the inequality

$$\cos x<\frac{\sin x}{x}<1.$$

This will imply that

$$\lim_{h\to 0}\frac{\cos h-1}{h}=\lim_{h\to 0}\frac{\cos^2 h-1}{h(\cos h+1)}=\lim_{h\to 0}\frac{\sin h}{h}\frac{\sin h}{\cos h+1}=0.$$

Example 51

$$\lim_{x\to 0}\frac{\sin 2x}{\sin 3x}=\lim_{x\to 0}\frac{\sin 2x}{2x}\cdot\frac{3x}{\sin 3x}\cdot\frac{2x}{3x}=\frac{2}{3}.$$

Infinite Limits: Vertical Asymptote

Definition 6

$$\lim_{x\to a} f(x)=\infty$$

means that $f(x)$ can be arbitrarily large as x tends to a;

$$\lim_{x\to a} f(x)=-\infty$$

means that $f(x)$ can be arbitrarily large negative as x tends to a.

Example 52 $\lim_{x\to 0}\frac{1}{x^2}=\infty$, $\lim_{x\to 1}\frac{-1}{(x-1)^2}=-\infty$.

Definition 7 *The line $x=a$ is called a vertical asymptote of the curve $y=f(x)$ if at least one of the following statements is true:*

$$\lim_{x\to a^-} f(x)=\pm\infty,\ \lim_{x\to a^+} f(x)=\pm\infty, \lim_{x\to a} f(x)=\pm\infty.$$

Example 53 *Find VA:* $f(x)=\tan x$, $\ln x$.

Limits at Infinity, HA

Definition 8 *The line $y=L$ is called a horizontal asymptote of the curve $y=f(x)$ if either*

$$\lim_{x\to\infty} f(x)=L \text{ or } \lim_{x\to-\infty} f(x)=L.$$

Example 54 $f(x)=\frac{3x^2-x-1}{2x^2+3x}$ *has horizontal asymptote* $y=\frac{3}{2}$.

Example 55 $\lim_{x\to\infty}\frac{a_nx^n+a_{n-1}x^{n-1}+\cdots+a_0}{b_mx^m+b_{m-1}x^{m-1}+\cdots+b_0}=\begin{cases}0, & \text{if } n<m;\\ \frac{a_n}{b_n}, & \text{if } n=m;\\ \pm\infty, & \text{if } n>m.\end{cases}.$

Example 56 $\lim\limits_{x\to\infty} \sin x$, $\lim\limits_{x\to\infty} \cos x$ *do not exist.*

Example 57 *Find the horizontal asymptotes of the function* $f(x) = e^x$.

Sol: $\lim_{x\to-\infty} e^x = 0$. Thus, HA: $\boxed{y = 0}$.

Example 58 *Find the horizontal asymptotes of the function*

$$f(x) = \sqrt{x^2+1} - x.$$

Solution:

$$\lim_{x\to\infty}(\sqrt{x^2+1} - x) = \lim_{x\to\infty} \frac{(\sqrt{x^2+1} - x)(\sqrt{x^2+1} + x)}{\sqrt{x^2+1} + x} = \lim_{x\to\infty} \frac{1}{\sqrt{x^2+1} + x} = 0.$$

Thus, HA: $\boxed{y = 0}$.

Example 59 *Find the horizontal asymptotes of the function*

$$f(x) = \sqrt{x^2+5x+1} - x.$$

Solution:

$$\begin{aligned}\lim_{x\to\infty}(\sqrt{x^2+5x+1} - x) &= \lim_{x\to\infty} \frac{(\sqrt{x^2+5x+1} - x)(\sqrt{x^2+5x+1} + x)}{\sqrt{x^2+5x+1} + x} \\ &= \lim_{x\to\infty} \frac{5x+1}{\sqrt{x^2+5x+1} + x} = \frac{5}{2}.\end{aligned}$$

Thus, HA: $\boxed{y = \frac{5}{2}}$.

Example 60 $y = \tan^{-1} x$ *has horizontal asymptotes* $y = \frac{\pi}{2}$ *and* $y = -\frac{\pi}{2}$.

Solution:

$$\lim_{x\to\infty} \tan^{-1} x = \frac{\pi}{2} \quad \textit{or} \quad \lim_{x\to-\infty} \tan^{-1} x = -\frac{\pi}{2}.$$

Infinite limits at ∞

The notation $\lim\limits_{x\to\infty} f(x) = \infty$ is used to indicate that the values of f(x) become large as x becomes large. Similar meanings are for

$$\lim_{x\to\infty} f(x) = -\infty, \lim_{x\to-\infty} f(x) = \infty, \lim_{x\to-\infty} f(x) = -\infty.$$

Example 61 $\lim\limits_{x\to\infty} x^5 = \infty$, $\lim\limits_{x\to-\infty} x^5 = -\infty$, $\lim\limits_{x\to\pm\infty}(x^3 - x^5) = \mp\infty$.

Example 62 $\lim_{x\to\infty} e^x = \infty$.

4.4 Continuity

Definition 9 *If* $\lim_{x\to a} f(x) = f(a)$*, then* $f(x)$ *is continuous at* $x = a$*, otherwise,* $f(x)$ *is discontinuous at* $x = a$*. If* $f(x)$ *is continuous at any point on an interval, then* $f(x)$ *is continuous on the interval. For the end points, we only need sided limits.*

Example 63 *Explore discontinuity from graph.*

Example 64 *Consider* $f(x) = \frac{x^2-2x+1}{x-1}$ *at* $x = 1$*.* $f(x)$ *is undefined at* $x = 1$*. But* $\lim_{x\to 1} f(x) = 0$*. So the discontinuous point* $x = 1$ *is* **removable** *if we define* $f(1) = 0$*.*

Example 65 *Determine the continuity of* $f(x) = \frac{|x|}{x}$*.*

Solution: $x = 0$ is not removable.

Definition 10 *If* $\lim_{x\to a^-} f(x) = f(a)$*, then* $f(x)$ *is continuous from the left at* $x = a$*; if* $\lim_{x\to a^+} f(x) = f(a)$*, then* $f(x)$ *is continuous from the right at* $x = a$*.*

Example 66 *Determine the left and right continuity at* $x = 0$*:*

$$f(x) = \begin{cases} \frac{|x|}{x}, & x \neq 0; \\ 1, & x = 0. \end{cases} .$$

Solution: continuous from right at $x = 0$, discontinuous from left at $x = 0$.

Theorem 4 *If* $f(x)$ *and* $g(x)$ *are continuous at* a*, then*

$$f \pm g, fg, cf\,(c \text{ is a constant}), \frac{f}{g}(\text{if } g(a) \neq 0)$$

are continuous.

Theorem 5 *Polynomials, rational functions, root functions, trig functions, inverse trig functions, exponential functions and logarithmic functions are continuous in their domain.*

Example 67 *Find* k *such that* $f(x) = \begin{cases} x^3 + kx^2 - 5x, & x > 2; \\ x - 4, & x \leq 2 \end{cases}$ *is continuous at any* $x \in \mathbb{R}$*.*

Solution:

$$8 + 4k - 10 = -2, \quad k = 0.$$

Example 68 *The greatest integer function [x].*

Theorem 6 *If* $\lim_{x\to a} g(x) = b$ *and* $f(x)$ *is continuous at* b*, then*

$$\lim_{x\to a} f(g(x)) = f(\lim_{x\to a} g(x)) = f(b).$$

Furthermore, if $g(x)$ *is continuous at* a*, and* $f(x)$ *is continuous at* $g(a)$*, then* $f(g(x))$ *is continuous at* a*.*

Example 69

$$\lim_{x\to 1} \arcsin(\frac{1-\sqrt{x}}{1-x}) = \arcsin(\lim_{x\to 1} \frac{1-\sqrt{x}}{1-x}) = \arcsin(\frac{1}{2}) = \frac{\pi}{6}.$$

4.5 Derivatives

Definition 11 *The derivative of the function* $y = f(x)$ *is the function* $f'(x)$*:*

$$f'(x) = \lim_{h\to 0} \frac{f(x+h) - f(x)}{h}.$$

Meaning of $f'(x)$:

- instantaneous rate of change of $f(x)$ at x, or
- rate of change of $f(x)$ at x, or
- the slope of the tangent line to the curve at x.

Example 70 *Find the slope and the equation of the tangent line to the curve*

$$y = f(x) = 3x^2 - 6x + 1$$

at the point $(2,1)$*. Sketch the curve.*

Solution: $f(2) = 1$.

$$m = \lim_{h\to 0} \frac{f(2+h) - f(2)}{h} = 6.$$

The tangent line is

$$y - 1 = 6(x-2), \Longrightarrow, y = 6x - 11.$$

Example 71 *Let* $f(x) = \sqrt{x-3}$*. Find* $f'(x)$ *and state the domains of* f *and* f'*.*

Solution:

$$f'(x) = \lim_{h\to 0} \frac{f(x+h) - f(x)}{h} = \lim_{h\to 0} \frac{\sqrt{x+h-3} - \sqrt{x-3}}{h}$$

$$= \lim_{h\to 0} \frac{(\sqrt{x+h-3} - \sqrt{x-3})(\sqrt{x+h-3} + \sqrt{x-3})}{h(\sqrt{x+h-3} + \sqrt{x-3})}$$

$$= \lim_{h\to 0} \frac{h}{h(\sqrt{x+h-3} + \sqrt{x-3})} = \lim_{h\to 0} \frac{1}{\sqrt{x+h-3} + \sqrt{x-3}} = \frac{1}{2\sqrt{x-3}}.$$

The domain of f: $x - 3 \geq 0$, $x \geq 3$.

The domain of f': $x - 3 \geq 0$ and $2\sqrt{x-3} \neq 0$, $x > 3$.

Example 72 *The volume of a sphere of radius r is given by*

$$V = \frac{4}{3}\pi r^3.$$

Calculate $\frac{dV}{dr}$ by definition. What's the meaning of this derivative?

Solution:

$$\frac{dV}{dr} = \lim_{h\to 0}\frac{V(r+h)-V(r)}{h} = \lim_{h\to 0}\frac{\frac{4}{3}\pi(r+h)^3 - \frac{4}{3}\pi r^3}{h}$$

$$= \frac{4}{3}\pi \lim_{h\to 0}\frac{(r+h)^3 - r^3}{h} = \frac{4}{3}\pi \lim_{h\to 0}\frac{3r^2h + 23rh^2 + h^3}{h} = 4\pi r^2.$$

The derivative is the surface area.

Example 73 *A spherical balloon is being inflated. Find the rate of change of the volume with respect to the radius when the radius is 2cm.*

Solution: Let r be the radius, $v(r)$ be the volume. From

$$v(r) = \frac{4}{3}\pi r^3$$

we have

$$\text{rate of change} = \lim_{\Delta r\to 0}\frac{\Delta v}{\Delta r} = \lim_{r\to 2}\frac{\frac{4}{3}\pi r^3 - \frac{4}{3}\pi 2^3}{r-2} = \lim_{r\to 2}\frac{\frac{4}{3}\pi(r-2)(r^2+2r+2^2)}{r-2} = 16\pi.$$

Definition 12 *The function f is differentiable at a if $f'(a)$ exists. It is differentiable on an interval if $f'(a)$ exists for any a on the interval.*

Theorem 7 *If a function is differentiable at $x = c$, then the function is continuous at $x = c$.*

Example 74 *$f(x) = |x|$ is not differentiable at $x = 0$.*

Solution:

$$f'(x) = \begin{cases} 1, & \text{if } x > 0; \\ -1, & \text{if } x < 0. \end{cases} \qquad f'(0)\nexists.$$

Remark. A point p in the domain $D(f)$ such that $f'(p) = 0$ or $f'(p)$ undefined is called a critical number (or critical point).

What Does f' Say About f?

Definition 13 *$y = f(x)$ is increasing on an interval I if $f(x_1) \leq f(x_2)$ for any $x_1 < x_2, x_1, x_2 \in I$; $y = f(x)$ is decreasing on an interval I if $f(x_1) \geq f(x_2)$ for any $x_1 < x_2, x_1, x_2 \in I$.*

INCREASING/DECREASING TEST (I/D TEST):

- If $f'(x) > 0$ on an interval, then f is increasing on that interval.
- If $f'(x) < 0$ on an interval, then f is decreasing on that interval.
- If $f'(x) = 0$ on an interval, then f is a constant on that interval.

Example 75 *Let $f(x) = x^4 - 4x^3 + 4x^2 + 4$. State all the intervals of increase and decrease.*

Solution:
(a) $f'(x) = 4x(x-1)x(x-2)$.

Definition 14 *Let $s = f(t)$ be position function.*

$$\textit{average velocity} = \frac{\textit{total distance}}{\textit{total time}} = \frac{\Delta s}{\Delta t}.$$

Instantaneous velocity, or velocity, or rate of change at $t = a$ is

$$v(a) = \lim_{h \to 0} \frac{f(a+h) - f(a)}{h}.$$

Example 76 *Consider the position function*

$$s = t^2 - 3t + 5.$$

Find the velocity at $t = 1$ and $t = 4$, interpret your results.

Solution:

$$v(1) = \lim_{t \to 1} \frac{t^2 - 3t + 5 - 3}{t - 1} = -1.$$

It means move backward.

$$v(4) = \lim_{t \to 1} \frac{t^2 - 3t + 5 - 9}{t - 4} = 5.$$

It means move forward.

5.1–5.6 Differentiation Rules

Derivatives of Polynomials and Exponential Functions

- Constant rule: If $f(x) = c$, then $f'(x) = 0$ or $\frac{d}{dx}(c) = 0$.
- Power Rule: If $f(x) = x^n$, n is any real number. Then $f'(x) = nx^{n-1}$.
- Constant multiple rule: $[cf(x)]' = cf'(x)$.
- Sum rule and difference rule: $[f(x) \pm g(x)]' = f'(x) \pm g'(x)$
- Derivative of polynomial: $[a_n x^n + a_{n-1}x^{n-1} + ... + a_1 x + a_0]' = a_n n x^{n-1} + a_{n-1}(n-1)x^{n-2} + ... + a_1$.
- Derivative of exponential function:

$$(e^x)' = e^x.$$

Example 77 *Let* $f(x) = a^x$, $a > 0$. *Then*

$$f'(x) = \lim_{h\to 0} \frac{a^{x+h} - a^x}{h} = a^x \lim_{h\to 0} \frac{a^h - 1}{h} = a^x f'(0).$$

Example 78 *Let* $f(x) = 4x^3 + 6x^2 - 23x + 7$. *Find the equation of the tangent line at* $(1, -6)$.

Solution: $f'(x) = 12x^2 + 12x - 23$. Let $y = mx + b$ be the tangent line. Then

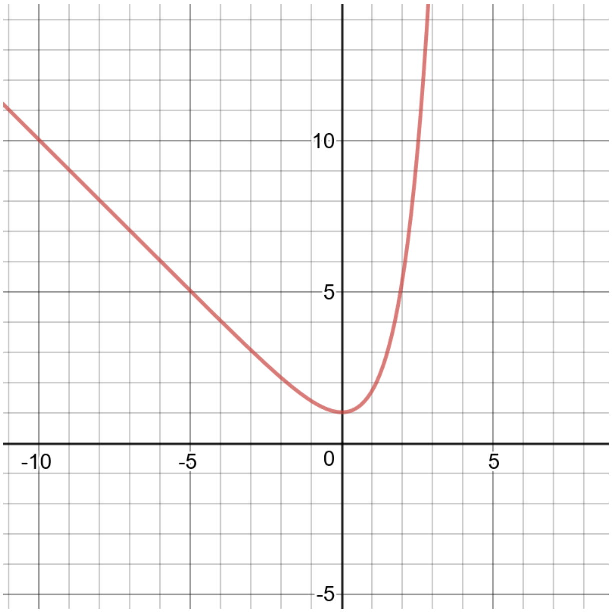
$$m = f'(1) = 1, \Rightarrow y = x + b.$$

Sub $(1, -6)$: $-6 = 1 + b, \Rightarrow b = -7, \Rightarrow y = x - 7$.

Example 79 *At what point(s) on the curve* $y = e^x - x$ *is the tangent line*

a) parallel to $y = 3x - 2$*?*

b) perpendicular to $y = -\frac{1}{2}x$*?*



Solution: (a) $(\ln 4, 4 - \ln 4)$.

(b) $(\ln 3, 3)$.

The product and quotient rules

- Product rule:
$$[f(x)g(x)]' = f'(x)g(x) + f(x)g'(x).$$
- Quotient rule:
$$\left(\frac{f(x)}{g(x)}\right)' = \frac{f'(x)g(x) - f(x)g'(x)}{g(x)^2}.$$

Example 80 *Let* $f(x) = \sqrt{x}e^x$*. Calculate* $f'(4)$*.*

Solution: $f'(x) = (\frac{1}{2\sqrt{x}} + \sqrt{x})e^x.$

Example 81 *Let* $f(x) = \frac{\sqrt{x}+x^2}{e^x+x}$*. Calculate* $f'(4)$*.*

Solution:

Example 82 *Let* $f(x) = \frac{x^3+4x^2}{x^5+x+1}$*. Calculate* $f'(1)$*.*

Example 83 *Let* $f(x) = \frac{x}{e^x}$*. Calculate* $f^{(n)}(x)$*.*

Solution:

$$\begin{aligned} f'(x) &= \frac{1-x}{e^x}, \\ f''(x) &= \frac{-(2-x)}{e^x}, \\ f'''(x) &= \frac{(3-x)}{e^x}, \\ &\vdots \\ f^{(n)}(x) &= \frac{(-1)^{n+1}(n-x)}{e^x}. \end{aligned}$$

Example 84 *At what point(s) on the curve* $y = \frac{x^2-4}{x+1}$ *is the tangent line*

a) parallel to $y = 3x$*?*

b) perpendicular to $y = -0.5x$*?*

Solution: By quotient rule,

$$y' = \frac{(x^2-4)'(x+1) - (x^2-4)(x+1)'}{(x+1)^2} = \frac{2x(x+1) - (x^2-4)1}{(x+1)^2} = \frac{x^2+2x+4}{(x+1)^2}.$$

a) Let $y' = 3 \Rightarrow \frac{x^2+2x+4}{(x+1)^2} = 3 \Rightarrow 2x^2 + 4x - 1 = 0 \Rightarrow x = -1 \pm \frac{\sqrt{6}}{2}$.

b) $(-0.5)y' = -1 \Rightarrow -0.5\frac{x^2+2x+4}{(x+1)^2} = -1 \Rightarrow x^2 + 2x - 2 = 0 \Rightarrow x = -1 \pm \frac{\sqrt{3}}{2}$.

The chain rule

- Chain Rule:

$$[f(g(x))]' = f'(g(x))g'(x), \quad \frac{df(g(x))}{dx} = \frac{df(v)}{dv} \cdot \frac{dg(x)}{dx}, v = g(x), \quad \frac{dy}{dx} = \frac{dy}{du} \cdot \frac{du}{dx}.$$

- General Power Rule:

$$[u(x)^n]' = nu^{n-1}u'(x).$$

Example 85 *Let* $f(x) = (x^2 - x - 1)^{100}$*. Calculate* $f'(x)$*.*

Solution: $f'(x) = 100(x^2 - x - 1)^{99}(x^2 - x - 1)' = 100(x^2 - x - 1)^{99}(2x - 1)$.

Example 86 *Let* $h(x) = g(f(x))$*, where* $f'(2) = 3$*,* $f(2) = 4$*,* $g'(3) = -5$*,* $g(4) = 8$*,* $g'(4) = 7$*. Find* $h'(2)$*.*

Solution: $h'(x) = g'(f(x))f'(x) \Rightarrow h'(2) = g'(f(2))f'(2) = g'(4)(3) = 7(3) = 21$.

Example 87 *Let* $y = \sqrt{x + \sqrt{x^2 + x}}$*. Calculate* y'*.*

Solution:

$$y' = \frac{1}{2} \frac{1}{\sqrt{x + \sqrt{x^2 + x}}} (x + \sqrt{x^2 + x})'$$

$$= \frac{1}{2\sqrt{x + \sqrt{x^2 + x}}} \left(1 + \frac{1}{2} \frac{1}{\sqrt{x^2 + x}} (x^2 + x)'\right) = \frac{1}{2\sqrt{x + \sqrt{x^2 + x}}} \left(1 + \frac{2x + 1}{2\sqrt{x^2 + x}}\right)$$

Derivative of exponential functions:

$$(a^x)' = a^x \ln a.$$

Proof.

$$(a^x)' = (e^{\ln a^x})' = (e^{x \ln a})' = (e^{x \ln a})(x \ln a)' = a^x \ln a.$$

Derivative of Logarithmic Function

By using the formula

$$\frac{df^{-1}(x)}{dx} = \frac{1}{f'(f^{-1}(x))},$$

We can get some special results:

- Derivatives of log functions:

$$\frac{d}{dx}(\ln x) = \frac{1}{x}, \quad (\ln f(x))' = \frac{f'(x)}{f(x)},$$

$$(\log_a |x|)' = \frac{1}{x \ln a}, \quad (\log_a f(x))' = \frac{f'(x)}{f(x) \ln a}, \ldots$$

Change base:

$$\log_a b = \frac{\log_c b}{\log_c a}.$$

Example 88 *Differentiate* $f(x) = \ln(x^2 + 1)$.

Solution:

$$f'(x) = \frac{2x}{x^2 + 1}.$$

Logarithmic differentiation

Example 89 *Differentiate* $y = \frac{(x^4+x+5)(5x^7-x^3+x+1)}{3x^2+2x+9}$.

Solution:

$\ln y = \ln(x^4 + x + 5) + \ln(5x^7 - x^3 + x + 1) - \ln(3x^2 + 2x + 9),$

$\frac{y'}{y} = \frac{4x^3 + 1}{x^4 + x + 5} + \frac{35x^6 - 3x^2}{5x^7 - x^3 + x + 1} + \frac{6x + 2}{3x^2 + 2x + 9}.$

Example 90 *Differentiate* $y = (x^2 + 1)^x$.

Solution:

$\ln y = x \ln(x^2 + 1),$

$\frac{y'}{y} = \ln(x^2 + 1) + \frac{2x^2}{x^2 + 1}.$

Number e

$$e = \lim_{x \to 0}(1 + x)^{1/x} = \lim_{x \to \infty}(1 + \frac{1}{x})^x.$$

Derivatives of Trigonometric Functions

Basic result:

$$\lim_{h\to 0}\frac{\sin h}{h}=1.$$

This will imply that

$$\lim_{h\to 0}\frac{\cos h-1}{h}=\lim_{h\to 0}\frac{\cos^2 h-1}{h(\cos h+1)}=\lim_{h\to 0}\frac{\sin h}{h}\frac{\sin h}{\cos h+1}=0.$$

Derivative of Trig Functions:

$$(\sin x)'=\cos x,\quad (\cos x)'=-\sin x,\quad (\tan x)'=\sec^2 x,$$

$$(\sec x)'=\sec x\tan x,\quad (\csc x)'=-\csc x\cot x,\quad (\cot x)'=-\csc^2 x.$$

Example 91 *Differentiate* $\csc x$, $\cot x$, $e^x\cos(x)$, $\frac{1+\cos x}{1+\sin x}$, $e^x\sin x$.

Solution: $(e^x\cos x)'=e^x(\cos x-\sin x)$.

Example 92 *Let* $y=\sin(x)$, *calculate* $y^{(10)}(x)$.

Example 93 *Given the position function* $s=f(t)=2\sin(t)$, *calculate the velocity and acceleration at* $t=\frac{\pi}{3}$.

Example 94 *Find the equation of the tangent line to the curve* $\sin(1-x)$ *at* $(1,0)$.

Example 95 *Differentiate*

$$f(x)=\sin x^2,\quad g(x)=\sin^2 x,\quad h(x)=e^{\sin x},\quad k(x)=\sin(\cos(\tan x)).$$

Solution: $f'(x)=2x\cos x^2, g'(x)=\sin 2x, h'(x)=(\cos x)e^{\sin x}$,
$k'(x)=-\sin(\tan x)\cos(\cos(\tan x))\sec^2 x$.

Example 96 *Differentiate* $y=(\sin x)^x$.

Implicit differentiation

Implicit Differentiation: Assume $f(x,y)=C$. To find y',

- consider x as an independent variable, y as a dependent variable;
- differentiate both sides with respect to x;
- isolate y'.

Example 97 *Find y' from $y^2+x^2=1$.*

Solution:

$$\frac{d}{dx}(y^2+x^2)=\frac{d1}{dx}, \Rightarrow 2yy'+2x=0, \Rightarrow y'=-\frac{x}{y}.$$

Example 98 *Let*

$$y^2+x^2=xy+3.$$

1) Find the equation of the tangent line to the curve at $(0,\sqrt{3})$.
2) Find all the points on the curve where the tangent line is either horizontal or vertical.

Solution: 1)

$$\frac{d}{dx}(y^2+x^2)=\frac{d}{dx}(xy+3), \Rightarrow 2yy'+2x=y+xy', \Rightarrow y'=\frac{y-2x}{2y-x}.$$

$$m=y'|_{(0,\sqrt{3})}=0.5, \Rightarrow y=0.5x+\sqrt{3}.$$

2) Horizontal tangent line: $y'=0 \Rightarrow y-2x=0 \Rightarrow x^2=1 \Rightarrow x=1, y=2$ or $x=-1, y=-2$.

Vertical tangent line: $y'=\infty \Rightarrow 2y-x=0 \Rightarrow y^2=1 \Rightarrow y=1, x=2$ or $y=-1, x=-2$.

Example 99 *Find $f'(x)$ from $f(x)\tan(xf(x)+x)=x+\sin x$. If $f(\pi)=-\frac{3}{4}$, what is $f'(\pi)$?*

Solution:

$$f'(x)=\frac{1+\cos x-f\sec^2(xf+x)(f+x)}{\tan(xf(x)+x)+xf\sec^2(xf+x)}.$$

Sub $(\pi,-3/4)$,

$$f'(\pi)=\frac{-9+12\pi}{16-12\pi}.$$

Derivatives of Inverse Trig Functions

$$\frac{d}{dx}\arcsin x = \frac{1}{\sqrt{1-x^2}}, \quad \frac{d}{dx}\arccos x = \frac{1}{-\sqrt{1-x^2}}, \quad \frac{d}{dx}\arctan x = \frac{1}{1+x^2},$$

$$\frac{d}{dx}\text{arcsec}x = \frac{1}{|x|\sqrt{x^2-1}}, \quad \frac{d}{dx}\text{arccsc}x = -\frac{1}{|x|\sqrt{x^2-1}}, \quad \frac{d}{dx}\text{arccot}x = -\frac{1}{1+x^2}.$$

Example 100 *Differentiate* $f(x) = \sin(\arctan 2x)$, $g(x) = \arcsin\left(\frac{b+a\cos x}{a+b\cos x}\right)$.

Solution: $f'(x) = \sin(\arctan 2x) \cdot \dfrac{2}{1+4x^2}$.

Example 101 *Differentiate* $y = \frac{(x^2+x+5)\arcsin x}{(x+1)^2}$.

Solution:

$$\begin{aligned} \ln y &= \ln(x^2+x+5) + \ln \arcsin x - 2\ln(x+1), \\ \frac{y'}{y} &= \frac{2x+1}{x^2+x+5} + \frac{1}{\arcsin x} \cdot \frac{1}{\sqrt{1-x^2}} - \frac{2}{x+1}, \\ y' &= \frac{(x^2+x+5)\arcsin x}{(x+1)^2}\left[\frac{2x+1}{x^2+x+5} + \frac{1}{\arcsin x} \cdot \frac{1}{\sqrt{1-x^2}} - \frac{2}{x+1}\right]. \end{aligned}$$

The Second Derivative, Concavity

Higher derivatives:

- Let $y = f(x)$. Then

$$y''(x) = f''(x) = \frac{d}{dx}\left(\frac{df}{dx}\right) = \frac{d}{dx}\left(\frac{dy}{dx}\right), \quad y^{(n)}(x) = f^{(n)}(x) = \frac{d}{dx}\left(\frac{dy^{(n-1)}}{dx}\right).$$

- If $s(t)$ is a position function, then the velocity is $v(t) = s'(t)$, acceleration is $a(t) = v'(t) = s''(t)$.

Example 102 *Let* $f(x) = 4x^3 + 6x^2 - 23x + 7$. *Then* $f''(x) = 24x + 12$, $f'''(x) = 24$ *and* $f^{(4)}(x) = 0$.

Example 103 *Let* $f(x) = (x^2 - x - 1)^{100}$. *Calculate* $f''(x)$.

Example 104 *The position of a particle is given by*

$$s = t^3 - 15t^2 + 63t, \quad t \geq 0$$

where s is measured in meters and t in seconds.

a) What is the initial position? initial velocity? initial acceleration?
b) Find the velocity after 1s and 4s.
c) When is the particle at rest?
d) When is the particle moving in the positive direction?
e) When is the acceleration 0?
f) Find the displacement and the velocity at that time from e).

Solution:

$$s = t^3 - 15t^2 + 63t, \Rightarrow s'(t) = 3t^2 - 30t + 63, \Rightarrow s''(t) = 6t - 30.$$

a) $s(0) = 0, v(0) = s'(0) = 63, a(0) = s''(0) = -30.$
b) $v(1) = s'(1) = 36$, $v(4) = s'(4) = -9$.
c) $s'(t) = 3t^2 - 30t + 63 = 0, \Rightarrow t = 3, 7$.
d) $s'(t) = 3t^2 - 30t + 63 > 0, \Rightarrow 0 < t < 3$, or $t > 7$.
e) $s'' = 0 \Rightarrow t = 5$.

Definition 15 *(CONCAVITY)If the graph of f lies above all of its tangents on an interval I (f' is increasing on I), it is called concave upward on I. If the graph of f lies below all of its tangents on I (f' is decreasing on I), it is called concave downward on I. If $f(x)$ changes concavity at p, then p is an inflection point, and $f''(p) = 0$ or undefined.*

CONCAVITY TEST: If $f''(x) > 0$ for all x in I, then the graph of f is concave upward on I. If $f''(x) < 0$ for all x in I, then the graph of f is concave downward on I.

Second Derivative Test: Let p be a critical number. If $f''(p) > 0$, then f has a local minimum at p; If $f''(p) < 0$, then f has a local maximum at p; If $f''(p) = 0$, then nothing.

Example 105 *Let $f(x) = x^4 - 4x^3 + 4x^2 + 4$.*

(c) Find all the local minimum points and all the local maximum points by Second Derivative Test.

(d) Find all the points of inflection.
(e) State intervals of concavity.
(f) Sketch the graph.

Example 106 *Consider the function*

$$f(x) = \frac{x}{x^2 - 1}.$$

Study the concavity and find all the points of inflection.

Solution: The domain of the function: $x \neq \pm 1$.

$$f'(x) = \frac{-1 - x^2}{(x^2 - 1)^2}, \quad f'' = \frac{2x(x^2 + 3)}{(x^2 - 1)^3}.$$

$$f''(x) = 0, \Rightarrow x = 0.$$

x	$-\infty < x < -1$	$-1 < x < 0$	$0 < x < 1$	$1 < x < \infty$
$f''(x)$	-	+	-	+
$f(x)$	concave down	concave up	concave down	concave up

Example 107 *Study the concavity of* $f(x) = x^{2x}$.

Solution: $f''(x) = 4(1 + x)e^{2x}$. POI is $(-1, -e^{-2})$.

6.5 Graphing Functions

The following checklist is intended as a guide to sketching a curve y = f(x) by hand.

Not every item is relevant to every function. For instance, a given curve might not have an asymptote or possess symmetry. However, the guidelines provide all the information you need to make a sketch that displays the most important aspects of the function.

- A. DOMAIN

- B. INTERCEPTS

- C. SYMMETRY

    1. EVEN FUNCTION: f(-x) = f(x) for all x in D. the curve is symmetric about the y-axis. This means that our work is cut in half.
    2. ODD FUNCTION: f(-x) = -f(x) for all x in D. the curve is symmetric about the origin. This means that our work is cut in half.
    3. PERIODIC FUNCTION: f(x + p) = f(x) for all x in D, where p is a positive constant. The smallest such number p is called the period.

- D. ASYMPTOTES

    - HORIZONTAL: $\lim_{x\to\pm\infty} f(x) = L$, then $y = L$ is a HA.
    - VERTICAL: $\lim_{x\to a^{\pm}} f(x) = \pm\infty$, then $x = a$ is a VA.
    - SLANT: If $\lim_{x\to\infty}[f(x) - (mx + b] = 0$, y =mx + b is called a slant asymptote.

- E. INTERVALS OF INCREASE OR DECREASE: use I/D Test.

- F. LOCAL MAXIMUM AND MINIMUM VALUES: First Derivative Test or Second Derivative Test.

- G. CONCAVITY AND POINTS OF INFLECTION

Example 108 *Use the first and second derivatives of $f(x) = e^{1/x}$, together with asymptotes, to sketch its graph.*

Solution: Notice that the domain of f is $\{x|x \neq 0\}$. So, we check for vertical asymptotes by computing the left and right limits as $x \to 0$.

$$\lim_{x\to 0^+} e^{1/x} = \infty, \quad \lim_{x\to 0^-} e^{1/x} = 0.$$

This shows that x = 0 is a vertical asymptote.

$$\lim_{x\to\pm\infty} e^{1/x} = 1,$$

this shows that y = 1 is a horizontal asymptote.

The Chain Rule gives:

$$f'(x) = -\frac{e^{1/x}}{x^2},$$

we have $f'(x) < 0$ for all $x \neq 0$. Thus, f is decreasing on $(-\infty, 0)$ and on $(0, \infty)$. There is no critical number. So, the function has no maximum or minimum.

$$f''(x) = \frac{e^{1/x}(2x+1)}{x^4},$$

$f''(x) > 0$ when $x > -1/2 (x \neq 0)$, and $f''(x) < 0$ when $x < -1/2$. So, the curve is concave downward on $(-\infty, -1/2)$ and concave upward on $(-1/2, 0)$ and on $(0, \infty)$.

The inflection point is $(-1/2, e^{-2})$.

Example 109 *Sketch the following functions:*

$$f(x) = x^4 + 8x^3 + 18x^2 + 1,$$

$$g(x) = \frac{2x^2}{x^2 - 4},$$
$$h(x) = xe^x,$$
$$k(x) = \frac{\ln x}{x^2}.$$

Example 110 *Suppose a function* $y = f(x), -\infty < x < \infty$, *is continuous, with continuous first and second derivatives. Assume it satisfies the following conditions:*

1. $f'(x) < 0$ *when* $x < 0$, *and* $f'(x) > 0$ *when* $x > 0$
2. $f''(x) < 0$ *when* $x < -2$, *and* $f''(x) > 0$ *when* $x > -2$
3. $\lim_{x \to \infty} f(x) = \infty, \lim_{x \to -\infty} f(x) = 2.$
4. $f(0) = -3, f(-2) = -1,\ f(-2.5) = 0,\ f(4) = 0,\ f(5) = 2.$

(a) *Where is the graph of* $f(x)$ *decreasing?*
(b) *Where is the graph of* $f(x)$ *concave up? Any point of inflection?*
(c) *Where does* $f(x)$ *attain a local maximum or minimum?*
(d) *What are the asymptotes of* f*?*
(e) *Sketch the graph of the function* $y = f(x)$.

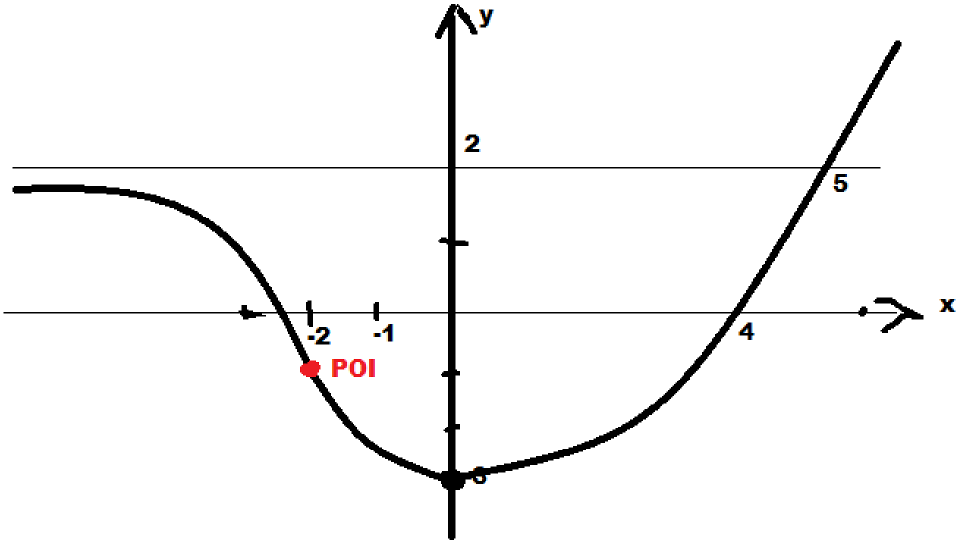
Solution: (a) decreasing until 0, then increasing

(b) concave down until -2, then concave up

(c) by the first derivative test (and continuity of f), $(0, -3)$ is a local minimum; this is supported by the second derivative test

(d) there is a horizontal asymptote as $x \to -\infty$.

(e) starting at $-\infty$, we are at an asymptote and decreasing and concave down, until $x = -2$, where we have a point of inflection at $(-2, -1)$, becoming concave up and touching a minimum at $(0, -3)$, then concave up and increasing all the way to ∞.



6.1-6.2 Applications of Derivatives

Maximum and Minimum Values

- Absolute (Global) Maximum and Minimum: $f(x)$ has a Global (Absolute) Maximum at p if $f(p) \geq f(x)$ for all x in the domain; $f(x)$ has a Global (Absolute) Minimum at p if $f(p) \leq f(x)$ for all x in the domain;
- Local (or relative) extrema: $f(x)$ has a local minimum at p if $f(p) \leq f(x)$ for points x near p; $f(x)$ has a local maximum at p if $f(p) \geq f(x)$ for points x near p;
- Critical number: A point p in the domain such that $f'(p) = 0$ or $f'(p)$ undefined is called a critical number, $(p, f(p))$ is a critical point, $f(p)$ is a critical value.

EXTREME VALUE THEOREM: If f(x) is continuous on a closed interval [a, b], then f attains an absolute maximum value f(c) and an absolute minimum value f(d) at some numbers c and d in [a, b].
FERMAT's THEOREM: If f has a local maximum or minimum at c, and if $f'(c)$ exists, then $f'(c) = 0$.

Definition 16 *Let $c \in D(f)$. If $f'(c) = 0$ or $f'(c)$ is undefined, then c is called a critical number (or critical point).*

Example 111 *Find the critical numbers of $f(x) = x^{3/5}(4 - x)$.*

Solution: 3/2 and 0.

First Derivative Test: Let c be a critical number. If f' changes from - to + at c, then f has a local minimum at c; If f' changes from + to - at c, then f has a local maximum at c.

Second Derivative Test: Let c be a critical number. If $f''(c) > 0$, then f has a local minimum at c; If $f''(c) < 0$ changes from + to - at c, then f has a local maximum at c; If $f''(c) = 0$, then the test provides no answer, go back to the first derivative test.

Example 112 *The Tradeoff between Medication and Site Effects: Suppose that a patient is given a dosage x of some medication, and the probability of a cure is*

$$P(x) = \frac{\sqrt{x}}{1+x}.$$

(a) Find the domain and the critical numbers.

(b) State the intervals of increase and decrease.

(c) Find the local maximum .

Solution: (a) Domain is $x > 0$. By the quotient rule,

$$P'(x) = \frac{1-x}{2\sqrt{x}(1+x)^2}.$$

Only critical number is $x = 1$.

(b) When $0 < x < 1$, $P'(x) > 0$, $P(x)$ is increasing; When $x > 1$, $P'(x) < 0$, $P(x)$ is decreasing.

(c) By the first derivative test, $P(1) = \frac{1}{2}$ is a local max.

Example 113 *Spread of a Pollutant: The concentration of a pollutant (measured in ppm, parts per million) at a fixed location x units from the source, is given by*

$$c(t) = \frac{N}{\sqrt{4\pi kt}} e^{-x^2/4kt},$$

where $N, k, t > 0$. When does the pollution reach its max?

Solution:

$$c'(t) = \frac{N(x^2 - 2kt)}{4kt^2\sqrt{4\pi kt}} e^{-x^2/4kt}.$$

The critical number is $t = \frac{x^2}{2k}$. When $0 < t < \frac{x^2}{2k}$, $c'(t) > 0$, $c(t)$ is increasing; When $t > \frac{x^2}{2k}$, $c'(t) < 0$, $c(t)$ is decreasing. By the first derivative test, the local max is

$$c(\frac{x^2}{2k}) = \frac{N}{x\sqrt{2\pi}} e^{-1/2}.$$

Example 114 *Let $g(x) = x + 2\sin x$, $0 \le x \le 2\pi$.*

(a) Find all the critical numbers.

(b) State all the intervals of increase and decrease.

(c) Find all the local minimum points and all the local maximum points.

Solution:

(a) $g'(x) = 1 + 2\cos x$, $g'(x) = 0 \Rightarrow x = 2\pi/3, 4\pi/3$.

(b) Look at the following table

x	$0 < x < 2\pi/3$	$2\pi/3 < x < 4\pi/3$	$4\pi/3 < x < 2\pi$
$f'(x)$	+	-	+
$f(x)$	increase	decrease	increase

Therefore,

The intervals of increase: $0 < x < 2\pi/3$, $4\pi/3 < x < 2\pi$.

The intervals of decrease: $2\pi/3 < x < 4\pi/3$

(c) Note that at $x = 2\pi/3$, $f'(x)$ changes from + to -; at $x = 4\pi/3$, $f'(x)$ changes from - to +. By the First Derivative Test, $f(x)$ has a local maximum at $x = 2\pi/3$ and a local minimum at $x = 4\pi/3$.

Absolute max and min, CLOSED INTERVAL METHOD: To find a global maximum or minimum for $f(x)$ on a closed interval $[a, b]$:

1. Find all the critical numbers, e.g., x_1, ..., x_n.
2. global minimum = $\min\{f(x_1), ..., f(x_n), f(a), f(b)\}$;

   global maximum = $\max\{f(x_1), ..., f(x_n), f(a), f(b)\}$.

Example 115 *Find the global maximum and minimum of the function*

$$f(x) = 2x^3 - 3x^2 - 12x + 7, \quad [-2, 0].$$

Solution:

Step 1) $f'(x) = 6x^2 - 6x - 12$, $f'(x) = 0 \Rightarrow x = -1, 2$, $f'(x)$ is defined anywhere. Hence $x = -1$ is the only one critical number in $(-2, 0)$.

Step 2) global minimum = $\min\{f(-2), f(-1), f(0)\} = \min\{3, 14, 7\} = 3$;

global maximum = $\max\{f(-2), f(-1), f(0)\} = \max\{3, 14, 7\} = 14$.

Example 116 *An open cylinder container has surface area $3\pi ft^2$. What dimensions will maximize the volume?*

Solution: Let r be the radius of the base, h be the height, A surface area, V the volume. Then

$$V = \pi r^2 h,$$

$$A = 2\pi rh + \pi r^2 = 3\pi, h = \frac{3 - r^2}{2r}.$$

$$V(r) = \frac{\pi}{2}(3r - r^3).$$

$$V'(r) = \frac{3\pi}{2}(1 - r^2).$$

$$V'(r) = 0 \Rightarrow r = 1.$$

$$V'(r) = -3\pi r < 0.$$

Thus $V(r)$ is concave down for all $r > 0$. Hence $V(r)$ is a global max at $r = 1$, $h = 1$.

Example 117 *Strength of Bones: The total mass of a bone and the mass of the marrow can be modeled by*

$$f(m) = c(2 - m^2)(1 - m^4)^{-2/3}, \quad 0 \leq m \leq 1,$$

where $m = 0$ characterizes a solid bone, and $m = 1$ describes a bone that is all marrow, m represents marrow cavity radius. When will the total mass reaches the minimum?

Solution:

$$f'(m) = -\frac{2}{3}cm(1 - m^4)^{-5/3}(m^4 - 8m^2 + 3).$$

Only critical number within the domain is $m = 0.628$, which gives min for $f(m)$.

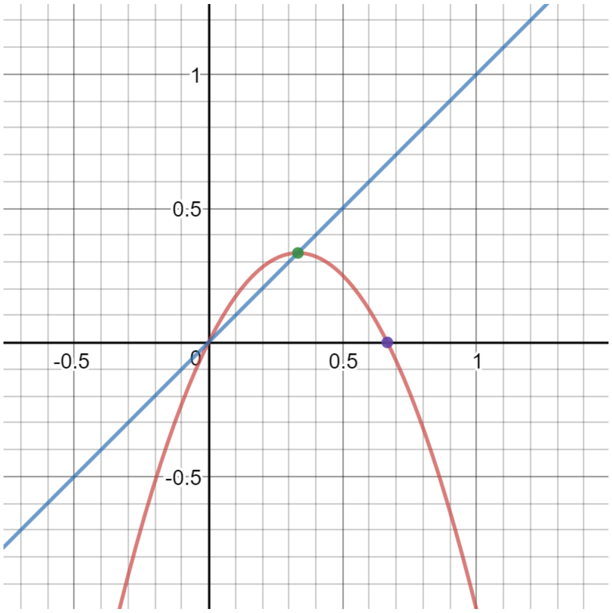
Example 118 *Sunfish population in Rideau Lake grows logistically and is harvested at a rate proportional to its population size:*

$$x_{t+1} = rx_t(1 - x_t) - hx_t,$$

where x_t represents sunfish population at time t, $r > 0$ is the growth rate, $h > 0$ is the harvesting rate, which reduces the growth rate of the fish by some amount.

(a) Find the fixed points (equilibria, or steady-states).

(b) At a fixed point x^*, the yield is given by $Y(h) = hx^*$. Find the maximum yield with $r = 2.5$.



Solution: (a):

$$x^* = rx^*(1-x^*) - hx^*, \Rightarrow x^* = 0, \frac{r-1-h}{r}.$$

(b): At $x^* = \frac{r-1-h}{r} = \frac{1.5-h}{2.5}$,

$$Y(h) = \frac{h(1.5-h)}{2.5}.$$

$Y'(h) = 0, \Rightarrow h = \frac{3}{4} = 0.75$. It is parabola, has global maximum at the critical number $h = 0.75 = \frac{3}{4}$: $Y(3/4) = 9/40$, which is the maximum yield.

6.4 L'Hospital's Rule

In this section, we are going to deal with the limit with the form:

$$\frac{0}{0},\quad \frac{\infty}{\infty},\quad 1^{\infty},\quad 0\cdot\infty,\quad 0^0,\ldots$$

L'Hospital's rule: If $\frac{f(x)}{g(x)}$ becomes $\frac{0}{0}$ or $\frac{\infty}{\infty}$ as $x\to x_0$, where x_0 is finite or ∞, then

$$\lim_{x\to x_0}\frac{f(x)}{g(x)}=\lim_{x\to x_0}\frac{f'(x)}{g'(x)}.$$

Remark. $x\to x_0$ can be replaced by any of the symbols $x\to x_0^+$, $x\to x_0^-$, $x\to\infty$, or $x\to-\infty$.

Example 119

$$\lim_{x\to 1}\frac{x^{2017}-5x^2+4}{x^{2018}+5x^3-6},\quad \lim_{x\to\infty}\frac{3x^2+2x-5}{2x^2+x+1}.$$

Solution:

$$\lim_{x\to 1}\frac{x^{2017}-5x^2+4}{x^{2018}+5x^3-6}=\lim_{x\to 1}\frac{2017x^{2016}-10x}{2018x^{2017}+15x^2}=\frac{2007}{2033}.$$

$$\lim_{x\to\infty}\frac{3x^2+2x-5}{2x^2+x+1}=\lim_{x\to\infty}\frac{6x+2}{4x+1}=\lim_{x\to\infty}\frac{6}{4}=\frac{3}{2}.$$

Example 120 *Calculate*

$$\lim_{x\to 0}\frac{\sin x}{x},\quad \lim_{x\to 0}\frac{1-\cos 3x}{x^2}.$$

Solution:

$$\lim_{x\to 0}\frac{\sin x}{x}=1,$$

$$\lim_{x\to 0}\frac{1-\cos 3x}{x^2}=\lim_{x\to 0}\frac{3\sin 3x}{2x}=\lim_{x\to 0}\frac{9\cos 3x}{2}=\frac{9}{2}.$$

Example 121 *Calculate*

$$\lim_{x\to\infty}x^2e^{-x},\quad \lim_{x\to\infty}\left(1-\frac{1}{x}\right)^x.$$

Solution:

$$\lim_{x\to\infty} x^2e^{-x} = \lim_{x\to\infty} \frac{x^2}{e^x} = \lim_{x\to\infty} \frac{2x}{e^x} = \lim_{x\to\infty} \frac{2}{e^x} = \lim_{x\to\infty} \frac{0}{e^x} = 0.$$

To solve the second limit, let $y = \left(1 - \frac{1}{x}\right)^x$, then

$$\ln y = x \ln\left(1 - \frac{1}{x}\right) = \frac{\ln\left(1 - \frac{1}{x}\right)}{\frac{1}{x}}.$$

$$\lim_{x\to\infty} \ln y = \lim_{x\to\infty} \frac{\ln\left(1 - \frac{1}{x}\right)}{\frac{1}{x}} = \lim_{x\to\infty} \frac{\frac{1}{x^2}}{-\frac{1}{x^2}\left(1 - \frac{1}{x}\right)} = -1. \Rightarrow$$

$$\lim_{x\to\infty} y = \frac{1}{e}.$$

Example 122 *Calculate*

$$\lim_{x\to 0} x^{\sin x}.$$

Solution: Let $y = x^{\sin x}$, then

$$\ln y = \ln x^{\sin x} = \sin x \ln x.$$

$$\lim_{x\to 0} \ln y = \lim_{x\to 0} \sin x \ln x = \lim_{x\to 0} \frac{\ln x}{\frac{1}{\sin x}} = \lim_{x\to 0} \frac{\frac{1}{x}}{\frac{-\cos x}{\sin^2 x}} = -\lim_{x\to 0} \frac{\sin x}{x} \cdot \frac{\sin x}{\cos x} = 0. \Rightarrow$$

$$\lim_{x\to 0} y = \lim_{x\to 0} e^{\ln y} = e^0 = 1.$$

Example 123 *Calculate*

(a) $\lim_{x\to\infty} \left(xe^{\frac{1}{x}} - x\right)$, *(b)* $\lim_{x\to\infty} \left(\sqrt{x^2 + x} - \sqrt{x^2 - 2x}\right)$.

Solution:

(a) $\lim_{x\to\infty} \left(xe^{\frac{1}{x}} - x\right) = \lim_{x\to\infty} \frac{e^{\frac{1}{x}} - 1}{\frac{1}{x}} = \lim_{t\to 0} \frac{e^t - 1}{t} = 1.$

(b) $\lim_{x\to\infty} \left(\sqrt{x^2 + x} - \sqrt{x^2 - 2x}\right) = \frac{3}{2}.$

5.7 Approximating Functions with Polynomials

LINEAR APPROXIMATIONS: we use the tangent line at $(a, f(a))$ as an approximation to the curve $y = f(x)$ when x is near a.

Definition 17 *The approximation*

$$f(x) \approx f(a) + f'(a)(x - a)$$

is called the linear approximation or tangent line approximation of f at the center (or, base point) a.

$$L(x) = f(a) + f'(a)(x - a)$$

is called the linearization of f at the center (or, base point) a.

Example 124 *Find the linearization of the function* $f(x) = \sqrt{x}$ *at* $a = 9$ *and use it to approximate the numbers* $\sqrt{9.01}$.

Solution:

$$\sqrt{x} \approx 3 + \frac{1}{6}(x - 9) = \frac{x}{6} + \frac{3}{2}, \qquad \sqrt{9.01} \approx 3 + \frac{0.01}{6}.$$

Example 125 *The linearization of the function* $f(x) = \sin x$ *at a = 0 is L(x) = x.*

Example 126 *Find the linearization of the function* $f(x) = e^{\sin 2x}$ *at the base point* $a = 1$.

Taylor polynomial Approximation: nth-degree Taylor polynomial of $f(x)$ at the base point (or centre) a is defined as

$$T_n(x) = f(a) + f'(a)(x - a) + \frac{f''(a)}{2!}(x - a)^2 + cdots + \frac{f^{(n)}}{n!}(x - a)^n.$$

Example 127 *Find the 4th-degree Taylor polynomial of* $f(x) = e^{3x}$ *at* $x = 0$ *and use it to approximate* $e^{0.1}$.

Solution:

$$T_4(x) = 1 + 3x + \frac{3^2}{2!}x^2 + \frac{3^3}{3!}x^3 + \frac{3^4}{4!}x^4.$$

$$e^{0.1} \doteq T_4(0.1) = 1.1051708.$$

The actual value of $e^{0.1}$ correcting to 7 decimals is 1.1051709.

Example 128 *Consider the function* $f(x) = 1 + \sin(2x - 2)$.
(a) Use a linear approximation of f *to estimate the value of* $f(0.9)$.
(b) Justify from the graph of f *why the approximation of* $f(0.9)$ *in (a) is below the actual value.*
(c) Use a Taylor polynomial of degree 3 *to approximate* $f(0.9)$.

Remark. This question did not specify the point a, we need to chose a value a, which is close to the point 0.9. So we'd like to take $a = 1$.

Solutions:

n	function	evaluated at $x = a = 1$
0	$f(x) = 1 + \sin(2x - 2)$	$f(1) = 1 + \sin(0) = 1$
1	$f'(x) = 2\cos(2x - 2)$	$f'(1) = 2$
2	$f''(x) = -4\sin(2x - 2)$	$f''(1) = 0$
3	$f'''(x) = -8\cos(2x - 2)$	$f'''(1) = -8$

So the linear approximation is

$$L(x) = T_1(x) = 1 + 2(x - 1)$$

(c):

$$T_3(x) = 1 + 2(x - 1) + 0(x - 1)^2 - 8/3!(x - 1)^3 = 1 + 2(x - 1) - \frac{4}{3}(x - 1)^3$$

Remark. Leave T_3 in factored form! By construction, $(x - 1)$ will be a nice little number that is fun to plug in, but if you multiply it out it will take all day. so

$$T_3(0.9) = 1 + 2(-0.1) - \frac{4}{3}(-0.1)^3 = 1 - 0.2 + 0.004/3 = 0.80133333333$$

whereas my calculator tells me that $f(0.9) = 0.8013306692$, so pretty good.

6.3 Reasoning about Functions

Intermediate Value Theorem (IVT): Let $f(x)$ be continuous on $[a, b]$ and K is a number between $f(a)$ and $f(b)$, then there is a number $c \in [a, b]$ such that $f(c) = K$.

Example 129 *Show that* $x^4 = 5x + 23$ *has solution in [2,4].*

Solution: Let $f(x) = x^4 - 5x - 23$.
The bisection method: Now that we know that there is a zero in a certain interval (from the IVT), we can use the IVT repeatedly (iteratively) to make the interval in which this zero occurs smaller and smaller.

MEAN VALUE THEOREM (MVT): Let $f(x)$ be a function that satisfies the following two hypotheses:

1. $f(x)$ is continuous on the closed interval [a, b]
2. $f(x)$ is differentiable on the open interval (a, b)

Then, there is a number c in (a, b) such that

$$f'(c) = \frac{f(b) - f(a)}{b - a}.$$

Example 130 *Sow that* $\sin x \leq x$ *for* $x \geq 0$*.*

Solution: Let $f(x) = x - \sin x$. By Mean Value Thm,

$$\frac{f(x) - f(0)}{x - 0} = f'(c) = 1 - \cos c \geq 0.$$

Thus

$$f(x) - f(0) \geq 0, f(x) \geq 0.$$

6.7-6.8 Stability of DTDS, Nonlinear Case

Recall: A linear DTDS $x_{t+1} = rx_t + c$ with $r \neq 1$ has exactly one fixed point $x^* = \frac{c}{1-r}$, and this point is stable when $|r| < 1$.

Theorem 8 *(Stability Theorem). Consider the DTDS*

$$x_{t+1} = f(x_t)$$

with an equilibrium x^*. If $|f'(x^*)| < 1$, then x^* is stable; If $|f'(x^*)| > 1$, then x^* is unstable.

Example 131 *Suppose the fraction x of mutant bacteria in a population of bacteria is given by the updating function*

$$f(x) = \frac{1.2x}{1.2x + 2(1-x)},$$

where 1.2 is the per capita production of the original type and 2 is the per capita production of the mutant type. Find the equilibria and analyze the stability.

Solution:

$$f(x) = x \Rightarrow x = 0, 1.$$

$$f'(x) = \frac{2.4}{(2-0.8x)^2}. \quad f'(0) = 0.6 < 1, f'(1) = 5/3 > 1.$$

Thus stable at $x = 0$ unstable at $x = 1$.

The Logistic Dynamical Systems

Consider the logistic dynamical system

$$N_{t+1} = rN_t(1 - \frac{N_t}{K}).$$

$$\text{per capita production} = r(1 - \frac{N_t}{K}),$$

where N represents population size, r is the greatest possible production, and K is the capacity. Let

$$x_t = \frac{N_t}{K},$$

then

$$x_{t+1} = rx_t(1 - x_t).$$

The equilibria are $x^* = 0, 1 - \frac{1}{r}$.

r	x^*	stability
0.5	0	stable
1.5	0	unstable
1.5	1/3	stable
2.5	0	unstable
2.5	0.6	stable
3.5	0	unstable
3.5	5/7	unstable

Example 132 *Concerning the above logistic dynamical system,* $f'(0) = r$, $f'(1 - 1/r) = 2 - r$.

Example 133 *Consider a population that grows according to the Beverton-Holt updating function and is harvested according to a linear rate* h. *The number of individuals of the species satisfies the DTDS*

$$x_{t+1} = \frac{4x_t}{1 + x_t} - hx_t, \quad t = 0, 1, 2, ...$$

(a) Find the fixed points of this DTDS.

(b) For which values of h *is there a positive fixed point?*

(c) Which harvesting rate maximizes the number of individuals harvested at the fixed point?

(d) Is the fixed point with the value of h *from part (c) stable? [If you did not get the answer to part (c), use* $h = 0.5$*]*

Solution: (a): $x^* = 0, \frac{3-h}{1+h}$.

(b) $h < 3$.

(c) The number of individuals harvested at the fixed point is

$$S(h) = hx^* = \frac{h(3 - h)}{1 + h}.$$

$S'(h) = \frac{4-(1+h)^2}{(1+h)^2}$. $S'(h) = 0, h = 1$. Note that $S'(h) > 0$ when $h < 1$ and $S'(h) < 0$ when $h > 1$, $S(1) = 1$ is max.

(d) The updating function is

$$f(x) = \frac{4x}{1 + x} - hx = \frac{4x}{1 + x} - x.$$

$f'(x) = \frac{4}{(1+x)^2} - 1, \Rightarrow |f'(0)| = 3 > 0; |f'(1)| = 0 < 1$. Therefore table at $x^* = 1$, unstable at $x^* = 0$.

6.6 Newton's Method

Sometimes we are presented with a problem which cannot be solved by simple algebraic means. For instance, if we needed to find the roots of the polynomial

$$x^3 - x + 1 = 0,$$

we would find that the tried and true techniques just wouldn't work. However, we will see that calculus gives us a way of finding approximate solutions.

Newton's method: To approximate solutions of the equation $f(x) = 0$, start from x_1, we have approximate solutions

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}$$

provided we have started with a good value for x_1 , this will produce approximate solutions to any degree of accuracy.

Example 134 *Find the roots* $\sqrt[6]{2}$ *by Newton's method, correct to 8 decimals.*

Solution: Let $f(x) = x^6 - 2$. Then $\sqrt[6]{2}$ is a solution of the equation $f(x) = 0$. Note that $f(2) = 62 > 0$ and $f(0) = -2 < 0$. This tells us that the root is between 0 and 2. So we chose $x_1 = 1$ for our initial guess.

$$x_{n+1} = x_n - \frac{x_n^6 - 2}{6x_n^5} = \frac{5x_n^6 + 2}{6x_n^5}.$$

With our initial guess of $x_1 = 1$, we can produce the following values:

x_0	1
x_1	1.16666667
x_2	1.12644368
x_3	1.12249707
x_4	1.12246205
x_5	1.12246205

Notice how the values for x_n become closer and closer to the same value. This means that we have found the approximate solution to 8 decimal places.

Example 135 *Find the roots of the polynomial* $f(x) = x^3 - x + 1 = 0$ *by Newton's method with* $x_0 = -1$*, correct to 6 decimals.*

Solution: Note that $f(-2) = -5$ and $f(0) = 1$. This tells us that the root is between -2 and 0. So we chose $x_1 = -0$ for our initial guess.

$$x_{n+1} = x_n - \frac{x_n^3 - x_n + 1}{3x_n^2 - 1} = \frac{2x_n^3 - 1}{3x_n^2 - 1}.$$

With our initial guess of $x_0 = -1$, we can produce the following values:

x_0	-1
x_1	-1.500000
x_2	-1.347826
x_3	-1.325200
x_4	-1.324718
x_5	-1.324717
x_6	-1.324717
x_7	-1.324717

Notice how the values for x_n become closer and closer to the same value. This means that we have found the approximate solution to six decimal places. In fact, this was obtained after only five relatively painless steps.

Method 2: Bisector method

$f(-2) = -5$ and $f(0) = 1$.
$f(-2) = -5$ and $f(-1) = 1$.
$f(-1.5) = -0.875$ and $f(-1) = 1$.
$f(-1.25) = ...$

Example 136 *Using Newton's method to find equilibrium*

$$x_{t+1} = e^{-x_t}$$

Solution: Let

$$f(x) = x - e^{-x}$$

$$f(1) > 0, f(-1) < 0$$

So we take $x_0 = 0$.

x_0	0
x_1	0.5
x_2	0.5663
x_3	0.5671
x_4	0.5671
x_5	0.5671

7.1 Differential Equations

n-th order DE: $f(y^{(n)}, ..., y', y, t) = 0$.

- Pure-time differential equation: $\frac{df(t)}{dt} = F(t)$.
- Autonomous differential equations: $\frac{df(t)}{dt} = F(f)$.
- Non-autonomous, non-pure-time differential equations: $\frac{df(t)}{dt} = F(f, t)$.

Some basic models:

- Exponential model: The rate of change of population growth is proportional to population size:
  $$P'(t) = rP,$$
  where r is a constant.
- Logistic model:
  $$P'(t) = rP(1 - \frac{P}{K}),$$
  where $r > 0$ is the relative growth rate, $L > 0$ is the capacity.
- Newton's Law of Cooling:
  $$\frac{dT}{dt} = \alpha(A - T),$$
  where α is a positive constant, A is a constant.

Example 137 *Verify that*

$$f(t) = 1 - e^{-t}$$

is a solution of the following differential equation:

$$\frac{df(t)}{dt} = e^{-t}, \quad f(0) = 0.$$

7.2 Antiderivatives

Definition 18 *A function $F(x)$ is called an antiderivative of $f(x)$ on an interval I if $F'(x) = f(x)$ for all x in I. We also call it the indefinite integral of $f(x)$.*

Some basic results:

function	antiderivative	formula
k	$kx + C$	$\int kdx = kx + C$
$x^n, n \neq -1$	$\frac{x^{n+1}}{n+1} + C$	$\int x^n dx = \frac{x^{n+1}}{n+1} + C;\ (n \neq -1)$
e^{kx}	$\frac{1}{k}e^{kx} + C$	$\int e^{kx}dx = \frac{1}{k}e^{kx} + C$
a^{kx}	$\frac{a^{kx}}{k \ln a} + C$	$\int a^{kx}dx = \frac{a^{kx}}{k \ln a} + C$
$\frac{1}{x}$	$\ln \lvert x \rvert + C$	$\int \frac{1}{x}dx = \ln \lvert x \rvert + C$
$\cos kx$	$\frac{1}{k}\sin kx + C$	$\int \cos kx dx = \frac{1}{k}\sin kx + C$
$\sin kx$	$-\frac{1}{k}\cos kx + C$	$\int \sin kx dx = -\frac{1}{k}\cos kx + C$
$\sec^2 kx$	$\frac{1}{k}\tan kx + C$	$\int \sec^2 kx dx = \frac{1}{k}\tan kx + C$
$\sec kx \tan kx$	$\frac{1}{k}\sec kx + C$	$\int \sec kx \tan kx dx = \frac{1}{k}\sec kx + C$
$\frac{1}{\sqrt{1-(kx)^2}}$	$\frac{1}{k}\arcsin kx + C$	$\int \frac{1}{\sqrt{1-(kx)^2}}dx = \frac{1}{k}\arcsin kx + C$
$\frac{1}{1+(kx)^2}$	$\frac{1}{k}\arctan kx + C$	$\int \frac{1}{1+(kx)^2}dx = \frac{1}{k}\arctan kx + C$
		$\int kf(x)dx = k\int f(x)dx$
		$\int [f(x) + g(x)]dx = \int f(x)dx + \int g(x)dx.$

Example 138 $\int \frac{x^2-1}{x^3}dx = \int (x^{-1} - x^{-3})dx = \ln \lvert x \rvert + \frac{1}{2x^2} + C.$

$\int \sin 4x + e^{5x}dx = -\frac{1}{4}\cos 4x + \frac{1}{5}e^{5x} + C.$

Example 139 *Find $f(x)$ such that*

$$f'(x) = \sin x + \frac{4x^2 - 22}{x^3}.$$

Solution:

$$f'(x) = \sin x + 4x^{-1} - 22x^{-3}, \Rightarrow f(x) = -\cos x + 4\ln \lvert x \rvert + 11x^{-2} + C.$$

Example 140 *Find $f(x)$ such that*

$$f'(x) = \sin x + \frac{4x^2 - 22}{x^3}, \quad f(1) = 3.$$

Solution:

$$f(x) = -\cos x + 4\ln|x| + 11x^{-2} + C.$$

$$f(1) = 3 \Rightarrow C = -8 + \cos 1, \Rightarrow f(x) = -\cos x + 4\ln|x| + 11x^{-2} - 8 + \cos 1.$$

Example 141 *The number of AIDS cases $A(t)$, where t is measured in years since 1981, is modeled by*

$$A'(t) = 523.8t^2, \quad A(0) = 340 \textit{ people.}$$

Find $A(t)$.

Solution:

$$A(t) = 174.6t^3 + 340.$$

7.3-7.4 Definite Integral and Area

Three ways to estimate the area of the region S bounded by the continuous function $y = f(x)$ (where $f(x) \geq 0$), $x = a$, $x = b$ and the x-axis:

We divide the interval $[a, b]$ into n equal parts with endpoints $x_0 = a$, $x_1 = a + \frac{b-a}{n}$, $x_2 = a + \frac{2(b-a)}{n}$,..., $x_n = a + \frac{n(b-a)}{n} = b$, $\Delta x = \frac{b-a}{n}$,

$$\begin{aligned} L_n &= \sum_{i=0}^{n-1} f(x_i)\Delta x = \left[f(x_0) + f(x_1) + ... + f(x_{n-1})\right]\Delta x, \\ R_n &= \sum_{i=1}^{n} f(x_i)\Delta x = \left[f(x_1) + ... + f(x_{n-1}) + f(x_n)\right]\Delta x, \\ M_n &= \sum_{i=1}^{n} f(\frac{x_{i-1}+x_i}{2})\Delta x = \left[f(\frac{x_0+x_1}{2}) + f(\frac{x_1+x_2}{2}) + ... + f(\frac{x_{n-1}+x_n}{2})\right]\Delta x. \end{aligned}$$

Here L_n is called Left-hand Sum, R_n is Right-hand Sum, M_n is called Midpoint Sum, or Midpoint Rule.

Definition 19 *The area under the curve $y = f(x) \geq 0$ between $x = a$ and $x = b$ is:*

$$Area = \lim_{n\to\infty} R_n = \lim_{n\to\infty} L_n = \lim_{n\to\infty} M_n.$$

Definition 20 *Definite integral = limit of Riemann sum:*

$$\int_a^b f(x)dx = \lim_{n\to\infty} GRS,$$

where

$$GRS(General\ Riemann\ Sum) = \sum_{i=1}^{n} f(c_i)\Delta x, \quad x_{i-1} \leq c_i \leq x_i, \Delta x = \frac{b-a}{n}.$$

The relation to area is:

$$\int_a^b f(x)dx = \text{area above } x\text{-axis - area below } x\text{-axis}.$$

Some basic properties about definite integral:

- $\int_a^b c dx = c(b-a)$;
- $\int_a^b f(x)dx = -\int_b^a f(x)dx$;
- $\int_a^a f(x)dx = 0$;
- $\int_a^c f(x)dx + \int_c^b f(x)dx = \int_a^b f(x)dx$;
- $\int_a^b [f(x) \pm g(x)]dx = \int_a^b f(x)dx \pm \int_a^b g(x)dx$;
- Constant multiple: $\int_a^b cf(x)dx = c\int_a^b f(x)dx$;

Example 142 *Let* $\int_1^5 f(x)\,dx = 3$, $\int_1^5 g(x)\,dx = 5$. *Calculate* $\int_1^5 [2f(x) - g(x) - 1]\,dx$.

Solution:

$$\int_1^5 [2f(x) - g(x) - 1]\,dx = 2\int_1^5 f(x)\,dx - \int_1^5 g(x)\,dx - \int_1^5 1\,dx$$
$$= 2(3) - 5 - 1(5-1) = -3.$$

The Fundamental Theorem of Calculus :

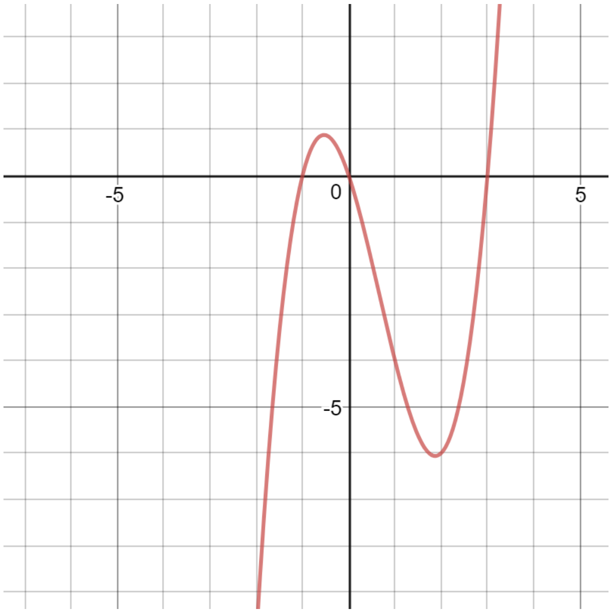
- If $F'(x) = f(x)$, then
$$\int_a^b f(x)dx = F(b) - F(a).$$

Example 143 *Calculate* $\int_0^2 3^t dt$.

Solution: Let $f(t) = 3^t$, then $F(t) = \frac{1}{\ln 3}3^t + C$.

$$\int_0^2 3^t dt = F(2) - F(0) = \frac{8}{\ln 3}.$$

Example 144 *Find the area of the region between the x-axis and the graph of* $f(x) = x^3 - 2x^2 - 3x$, $-1 \leq x \leq 3$.



Solution:

Step 1: Find zeros of $f(x)$: Let $f(x) = 0$, then $x = -1, 0, 3$;

Step 2:

$$\text{total area} = \int_{-1}^{0} f(x)dx + \int_{0}^{3} -f(x)dx$$

$$= \left(\frac{1}{4}x^4 - \frac{2}{3}x^3 - \frac{3}{2}x^2\right)\Bigg|_{-1}^{0} - \left(\frac{1}{4}x^4 - \frac{2}{3}x^3 - \frac{3}{2}x^2\right)\Bigg|_{0}^{3} = \frac{7}{12} + \frac{45}{4} = \frac{61}{6}.$$

7.5 Substitution and Integration by Parts

Substitution

- For indefinite integral: $\int f(g(x))g'(x)dx = \int f(u)du, \quad u = g(x)$. In the final result, we have to replace u by $g(x)$;
- For definite integral: $\int_a^b f(g(x))g'(x)dx = \int_{g(a)}^{g(b)} f(u)du$.

Example 145 *Evaluate*

$$\int (2x-1)(x^2-x)^{100}\,dx.$$

Solution: Let $u = x^2 - x$. Then $du = (2x-1)dx$. Thus

$$\int (2x-1)(x^2-x)^{100}\,dx = \int u^{100}du = \frac{u^{101}}{101} + C = \frac{(x^2-x)^{101}}{101} + C.$$

Example 146

$$\int x\sqrt{x^2+1}\,dx.$$

Solution: Let $u = x^2 + 1$. Then $du = (2x)dx$. Thus

$$\int x\sqrt{x^2+1}\,dx = \int \frac{1}{2}\sqrt{u}\,du = \frac{1}{2}\int u^{1/2}\,du = \frac{1}{2}\frac{u^{3/2}}{3/2} + C = \frac{(x^2+1)^{3/2}}{3} + C.$$

Example 147

$$\int_0^1 x\sqrt{x^2+1}\,dx.$$

Solution: Let $u = x^2 + 1$. Then $du = (2x)dx$, $x = 0 \leftrightarrow u = 1$, $x = 1 \leftrightarrow u = 2$. Thus

$$\int_0^1 x\sqrt{x^2+1}\,dx = \int_1^2 \frac{1}{2}\sqrt{u}\,du = \frac{1}{2}\int u^{1/2}\,du = \frac{1}{2}\left[\frac{u^{3/2}}{3/2}\right]_1^2 = \frac{2\sqrt{2}-1}{3}.$$

Example 148 *Find*

$$\int \frac{1}{e^{-x}+1}dx.$$

Solution:

$$\int \frac{1}{e^{-x}+1}dx = \int \frac{e^x}{1+e^x}dx.$$

Let $u = 1 + e^x$, then $du = e^x dx$.

$$\int \frac{e^x}{1+e^x}dx = \int \frac{1}{u}du = \ln|u| + C = \ln|1+e^x| + C = \ln(1+e^x) + C.$$

Example 149 *Evaluate*

$$\int x^2 e^{x^3+1}\, dx.$$

Solution: Let $u = x^3 + 1$, $du = 3x^2 dx$.

Example 150 *Evaluate*

$$\int \frac{x}{\sqrt{1-x}}\, dx.$$

Solution: Let $u = 1 - x$, $du = -dx$.

$$\int \frac{x}{\sqrt{1-x}}\, dx = \int \frac{1-u}{\sqrt{u}}\, (-du) = \int \left(-\frac{1}{\sqrt{u}} + \sqrt{u}\right) du = -2\sqrt{u} + \frac{2}{3}u^{3/2} + C$$

$$= -2\sqrt{1-x} + \frac{2}{3}(1-x)^{3/2} + C.$$

Example 151 *Calculate*

$$\int \tan x\, dx.$$

Solution: Note that $\tan x = \dfrac{\sin x}{\cos x}$, we let $u = \cos x$. Then $du = -\sin x\ dx$. Thus

$$\int \tan x\, dx = \int \frac{\sin x}{\cos x}\ dx = -\int \frac{1}{u}\ du = -\ln|u| + C = -\ln|\cos x| + C.$$

Example 152 *Evaluate*

$$\int \frac{\arcsin x}{\sqrt{1-x^2}}\, dx.$$

Solution: Let $u = \arcsin x$, $du = \frac{1}{\sqrt{1-x^2}}dx$,

$$\int \frac{\arcsin x}{\sqrt{1-x^2}}\, dx = \int u du.$$

.

Example 153 *Evaluate*

$$\int \frac{(\ln x)^{2017}}{x}\, dx.$$

Solution: Let $u = \ln x$, $du = \frac{1}{x}dx$,

$$\int \frac{(\ln x)^{2017}}{x}\, dx = \int u^{2017} du.$$

.

Integration by Parts

Integration by parts Formula:

$$\int u(x)v'(x)dx = u(x)v(x) - \int u'(x)v(x)dx, \quad \textit{or} \quad \int u(x)\, dv(x) = u(x)v(x) - \int v(x)\, du(x).$$

Example 154 *Evaluate*

$$\int x^2 e^{bx}\, dx, \quad b \neq 0.$$

Solution: Method 1:

$$\begin{aligned}
\int x^2 e^{bx}\, dx &= \int x^2\, d\left(\frac{1}{b}e^{bx}\right) = x^2\left(\frac{1}{b}e^{bx}\right) - \frac{1}{b}\int e^{bx}\, d(x^2) \\
&= \frac{1}{b}x^2 e^{bx} - \frac{2}{b}\int x e^{bx}\, dx = \frac{1}{b}x^2 e^{bx} - \frac{2}{b}\int x d\left(\frac{1}{b}e^{bx}\right) \\
&= \frac{1}{b}x^2 e^{bx} - \frac{2}{b}\left(x\frac{1}{b}e^{bx} - \frac{1}{b}\int e^{bx}\, dx\right) \\
&= \frac{1}{b}x^2 e^{bx} - \frac{2}{b^2}x e^{bx} + \frac{2}{b^2}\int e^{bx}\, dx \\
&= \frac{1}{b}x^2 e^{bx} - \frac{2}{b^2}x e^{bx} + \frac{2}{b^3}e^{bx} + c
\end{aligned}$$

Method 2:

$$\int x^2 e^{bx}\, dx = \int x^2\left(\frac{1}{b}e^{bx}\right)' dx = x^2\left(\frac{1}{b}e^{bx}\right) - \frac{1}{b}\int (x^2)' e^{bx}\, dx$$

$$= \frac{1}{b}x^2 e^{bx} - \frac{2}{b}\int x e^{bx}\, dx = \frac{1}{b}x^2 e^{bx} - \frac{2}{b}\int x\left(\frac{1}{b}e^{bx}\right)' dx$$

$$= \frac{1}{b}x^2e^{bx} - \frac{2}{b}\left(x\frac{1}{b}e^{bx} - \frac{1}{b}\int (x)'e^{bx}\,dx\right)$$

$$= \frac{1}{b}x^2e^{bx} - \frac{2}{b^2}xe^{bx} + \frac{2}{b^2}\int e^{bx}\,dx$$

$$= \frac{1}{b}x^2e^{bx} - \frac{2}{b^2}xe^{bx} + \frac{2}{b^3}e^{bx} + c$$

Method 3: Let $t = bx$. Then $dt = bdx$.

$$\int x^2e^{bx}\,dx = \frac{1}{b^3}\int t^2e^t\ dt = \frac{1}{b^3}\left(t^2e^t - 2te^t + 2e^t + c\right).$$

Remark. x^2 can be replaced by any polynomial.

Example 155 *Evaluate*

$$\int 4x^3 \ln x\,dx.$$

Solution: Integration by parts

$$\int 4x^3\ln x\,dx = \int \ln x\,dx^4 = x^4\ln x - \int x^4\,d(\ln x)$$

$$x^4\ln x - \int x^4\,\frac{dx}{x} = x^4\ln x - \int x^3\,dx$$

$$= x^4\ln x - \frac{1}{4}x^4 + C$$

Example 156 *Evaluate*

$$\int_0^1 \arctan x\,dx.$$

Solution: Step 1: Calculate

$$\int \arctan x\,dx = x\arctan x - \int \frac{x}{1+x^2}dx = x\arctan x - \frac{1}{2}\int\frac{d(1+x^2)}{1+x^2}$$

$$= x\arctan x - \frac{1}{2}\ln(1+x^2).$$

Step 2:

$$\int_0^1 \arctan x\,dx = [x\arctan x - \frac{1}{2}\ln(1+x^2)]|_0^1 = \arctan 1 - \frac{1}{2}\ln 2 = \frac{\pi}{4} - \frac{1}{2}\ln 2.$$

Example 157 *Evaluate*

$$\int e^{ax}\cos(bx)\,dx.$$

Example 158 *Evaluate*

$$\int x\sin(bx)\,dx.$$

Solution: Method 1:

$$\int x\sin(bx)\,dx = \int x\;d\left(-\frac{\cos bx}{b}\right) = x\left(-\frac{\cos bx}{b}\right) - \int\left(-\frac{\cos bx}{b}\right)\;dx$$

$$= -\frac{1}{b}x\cos bx + \frac{1}{b}\int\cos bx\;dx = -\frac{1}{b}x\cos bx + \frac{1}{b^2}\sin bx\;dx + c$$

Method 2:

$$\int x\sin(bx)\,dx = \int x\;\left(-\frac{\cos bx}{b}\right)'\;dx = x\left(-\frac{\cos bx}{b}\right) - \int\left(-\frac{\cos bx}{b}\right)(x)'\;dx$$

$$= -\frac{1}{b}x^3\cos bx + \frac{1}{b}\int\cos bx\;dx$$

Method 3: Let $t = bx$, then $dt = bdx$, $dx = \frac{1}{b}dt$.

$$\int x\sin(bx)\,dx = \frac{1}{b^2}\int t\sin t\;dt.$$

Example 159 *Find the function $f(x)$, such that*

$$f''(x) = 2\ln(x) + 2, \quad f(1) = 1, \quad f'(1) = 0.$$

Solution:

$$f'(x) = \int f''(x)dx = \int(2\ln(x)+2)(x)'dx = (2\ln(x)+2)x - \int(2\ln(x)+2)'(x)dx$$

$$= (2\ln(x)+2)x - \int 2dx = 2x\ln x + c.$$

$f'(1) = c = 0, \Rightarrow f'(x) = 2x\ln x.$

$$f(x) = \int f'(x)dx = \int 2x\ln(x)dx = \int(x^2)'\ln(x)dx = x^2\ln(x) - \int x^2[\ln(x)]'dx$$

$$= x^2\ln(x) - \int xdx = x^2\ln x - \frac{1}{2}x^2 + c.$$

$f(1) = -\frac{1}{2} + c = 1, \Rightarrow c = 1.5.$ Thus
$f(x) = x^2\ln x - \frac{1}{2}x^2 + 1.5.$