ECON 325
Concordia University
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Fall 2018, Assignment 3 - Solutions

1. (24 pts) You are given the following system,

$$\begin{aligned} x+y+z &= a \\ y+z &= ab \\ z &= abc \end{aligned}$$

where a, b and c are parameters.

(a) Express x and y in terms of a, b and c. (6 pts)

Since $z = abc$, using the $2nd$ equation: $y + z = ab \Rightarrow y = ab(1-c)$.
Also, using the $1st$ equation: $x + y + z = a \Rightarrow x = a - y - z \Rightarrow x = a(1-b)$.
Thus, $x = a(1-b)$, $y = ab(1-c)$ and $z = abc$.

(b) Find the partial derivatives of the three variables (x, y, and z) with respect to a, b and c and write their gradient vectors. (6 pts)

For $x = a(1-b)$, we have the following partial derivatives:
$\frac{\partial x}{\partial a} = (1-b)$, $\frac{\partial x}{\partial b} = -a$ and $\frac{\partial x}{\partial c} = 0$.
For $y = ab(1-c)$, we have the following partial derivatives:
$\frac{\partial y}{\partial a} = b(1-c)$, $\frac{\partial y}{\partial b} = a(1-c)$ and $\frac{\partial y}{\partial c} = -ab$.
For $z = abc$, we have the following partial derivatives:
$\frac{\partial z}{\partial a} = bc$, $\frac{\partial z}{\partial b} = ac$ and $\frac{\partial z}{\partial c} = ab$.
The gradient vectors are:
$\nabla x(a,b,c) = (1-b, -a, 0)$, $\nabla y(a,b,c) = (b(1-c), a(1-c), -ab)$ and $\nabla z(a,b,c) = (bc, ac, ab)$.

(c) Construct the Jacobian matrix and find the determinant. (6 pts)

The Jacobian matrix is,

$$J = \begin{bmatrix} \frac{\partial x}{\partial a} & \frac{\partial x}{\partial b} & \frac{\partial x}{\partial c} \\ \frac{\partial y}{\partial a} & \frac{\partial y}{\partial b} & \frac{\partial y}{\partial c} \\ \frac{\partial z}{\partial a} & \frac{\partial z}{\partial b} & \frac{\partial z}{\partial c} \end{bmatrix} = \begin{bmatrix} (1-b) & -a & 0 \\ b(1-c) & a(1-c) & -ab \\ bc & ac & ab \end{bmatrix}.$$

The Jacobian determinant is (expansion using the $1st$ row),

$$|J| = \begin{vmatrix} (1-b) & -a & 0 \\ b(1-c) & a(1-c) & -ab \\ bc & ac & ab \end{vmatrix} = (1-b)(-1)^2 \begin{vmatrix} a(1-c) & -ab \\ ac & ab \end{vmatrix} + (-a)(-1)^3 \begin{vmatrix} b(1-c) & -ab \\ bc & ab \end{vmatrix} + 0 =$$

$$(1-b)\left[a^2b(1-c) + a^2bc\right] + a\left[ab^2(1-c) + ab^2c\right] = a^2b(1-c) + a^2bc - a^2b^2(1-c) - a^2b^2c + a^2b^2(1-c) + a^2b^2c = a^2b(1-c) + a^2bc = a^2b.$$

(d) Under which condition(s) are the three functions functionally independent? (6 pts)

The three functions are functionally independent when $|J| \neq 0$. Thus $a^2b \neq 0 \Rightarrow a \neq 0$ and $b \neq 0$.

2. (24 pts) The system of the following equations describes the market for USB flash drives of brand "A".

$$\begin{aligned} Q_d &= \alpha - \beta P + \gamma S \\ Q_s &= -\delta + \kappa P - \lambda C \\ Q_d &= Q_s \end{aligned}$$

where α, β, γ, δ, κ and λ are positive parameters. Also, S is the price of substitutes for USB flash drives and C is the price of inputs used in producing USB flash drives. Both S and C are exogenous variables for this model.

(a) Find the equilibrium quantity P^* and price Q^* for the USB flash drives (both of them should be functions of the parameters and exogenous variables of the model). (6 pts)

From the equilibrium condition $Q_d = Q_s$ we derive the equilibrium price P^*. That is,
$Q_d = Q_s \Rightarrow \alpha - \beta P + \gamma S = -\delta + \kappa P - \lambda C \Rightarrow P^* = \frac{\alpha+\delta+\gamma S+\lambda C}{\beta+\kappa}$.
Substituting P^* into either Q_d or Q_s we derive the equilibrium quantity Q^*. That is,
$Q^* = \alpha - \beta\frac{\alpha+\delta+\gamma S+\lambda C}{\beta+\kappa} + \gamma S = \frac{\alpha(\beta+\kappa)}{\beta+\kappa} - \beta\frac{\alpha+\delta+\gamma S+\lambda C}{\beta+\kappa} + \frac{\gamma(\beta+\kappa)}{\beta+\kappa}S = \frac{\alpha(\beta+\kappa)-\beta(\alpha+\delta+\gamma S+\lambda C)+\gamma(\beta+\kappa)S}{\beta+\kappa} = \frac{\alpha\kappa-\beta(\delta+\lambda C)+\gamma\kappa S}{\beta+\kappa} \Rightarrow Q^* = \frac{\kappa(\alpha+\gamma S)-\beta(\delta+\lambda C)}{\beta+\kappa}$.
Therefore,

$$\begin{aligned} P^* &= \frac{\alpha + \delta + \gamma S + \lambda C}{\beta + \kappa}, \\ Q^* &= \frac{\kappa(\alpha + \gamma S) - \beta(\delta + \lambda C)}{\beta + \kappa}. \end{aligned}$$

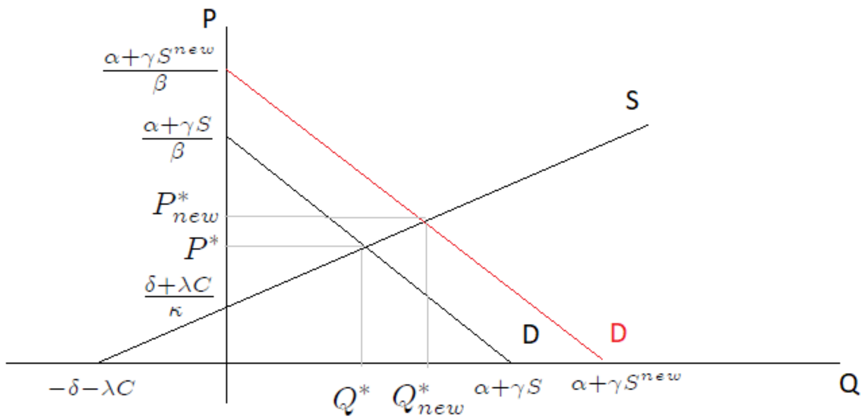
(b) Find the partial derivatives of P^* and Q^* with respect to S and C. (6 pts)

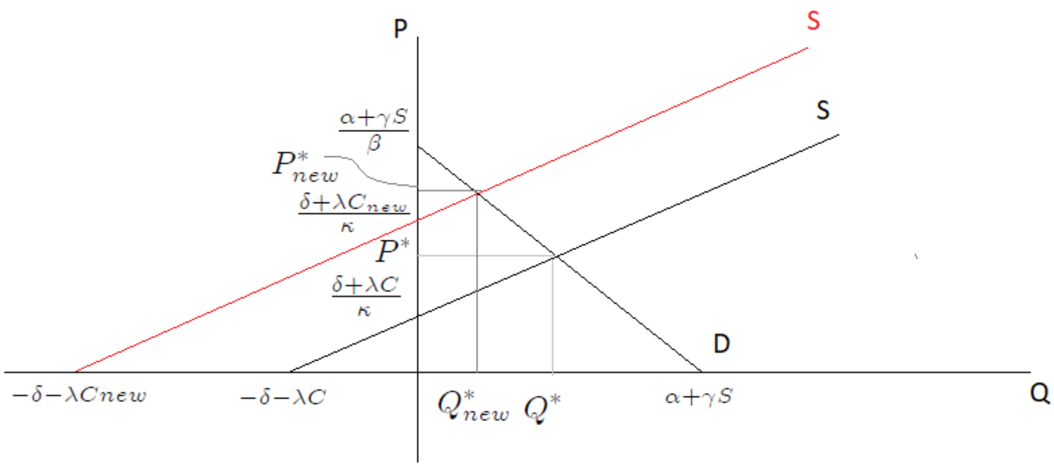
The partial derivatives of P^* with respect to S and C are the following:
$\frac{\partial P^*}{\partial S} = \frac{\gamma}{\beta+\kappa} > 0$ and $\frac{\partial P^*}{\partial C} = \frac{\lambda}{\beta+\kappa} > 0$.
The partial derivatives of Q^* with respect to S and C are the following:
$\frac{\partial Q^*}{\partial S} = \frac{\kappa\gamma}{\beta+\kappa} > 0$ and $\frac{\partial Q^*}{\partial C} = -\frac{\beta\lambda}{\beta+\kappa} < 0$.

(c) Show (algebraically and graphically) how an increase in the price of substitute goods, S, affects equilibrium quantity and price and how an increase in the price of inputs, C, affects equilibrium quantity and price. Do we expect these results? (6 pts)

Since $\frac{\partial Q^*}{\partial S} = \frac{\kappa\gamma}{\beta+\kappa} > 0$, an increase in the price of substitutes will increase equilibrium quantity.
Since $\frac{\partial P^*}{\partial S} = \frac{\gamma}{\beta+\kappa} > 0$, an increase in the price of substitutes will increase equilibrium price.
The results are in line with our expectations.
Since $\frac{\partial Q^*}{\partial C} = -\frac{\beta\lambda}{\beta+\kappa} < 0$, an increase in the price of inputs will reduce equilibrium quantity.
Since $\frac{\partial P^*}{\partial C} = \frac{\lambda}{\beta+\kappa} > 0$, an increase in the price of inputs will increase equilibrium price.
The results are in line with our expectations.

(d) Find the partial elasticity of the demand with respect to S (the price of substitutes) and the partial elasticity of the supply with respect to C (the price of inputs). Give an interpretation of the derived results. (6 pts)





The partial elasticity of the demand with respect to S is:
$\varepsilon_{Q_d s} = \frac{\partial Q_d}{\partial S}\frac{S}{Q_d} = \gamma\frac{S}{\alpha-\beta P+\gamma S} > 0$.

If the price of substitutes for USB flash drives is increased by, say, 1% (keeping constant the price of the USB flash drives), the quantity demanded for USB flash drives will increase by $\left(\gamma\frac{S}{\alpha-\beta P+\gamma S}\right)$%.

The partial elasticity of the supply with respect to C is:
$\varepsilon_{Q_s c} = \frac{\partial Q_s}{\partial C}\frac{C}{Q_s} = -\lambda\frac{C}{-\delta+\kappa P-\lambda C} < 0$.

If the price of inputs used in producing USB flash drives is increased by, say, 1% (keeping constant the price of the USB flash drives), the quantity supplied for USB flash drives will decrease by $\left(-\lambda\frac{C}{-\delta+\kappa P-\lambda C}\right)$%.

3. (24 pts) Suppose the aggregate output of an economy is given by,

$$Y = K^\alpha (AL)^{1-\alpha} \quad 0 < \alpha < 1$$

where K is the capital, L is the labour force and A is the effectiveness of labour. Capital, labour force and effectiveness of labour are all growing over time (t) according to:

$$K(t) = K_0 e^{mt},\ L(t) = L_0 e^{nt} \text{ and } A(t) = A_0 e^{pt}$$

(a) Find the proportional growth rates of capital, labour force and effectiveness of labour. (6 pts)

The proportional growth rate of capital is:
$\frac{dK(t)}{dt} = mK_0e^{mt} = mK(t)$.
So, capital is growing at constant proportional rate m.
The proportional growth rate of labour is:
$\frac{dL(t)}{dt} = nL_0e^{nt} = nL(t)$.
So, labour is growing at a constant proportional rate n.
The proportional growth rate of effectiveness of labour is:
$\frac{dA(t)}{dt} = pA_0e^{pt} = pA(t)$.
So, effectiveness of labour is growing at a constant proportional rate p.

(b) Find the growth rate of the aggregate output. (6 pts)

Since capital, labour and effectiveness of labour are all functions of time, time effects output through three channels.

Taking total derivative, we have:
$\frac{dY}{dt} = \frac{\partial Y}{\partial K}\frac{dK}{dt} + \frac{\partial Y}{\partial A}\frac{dA}{dt} + \frac{\partial Y}{\partial L}\frac{dL}{dt} = \alpha K^{\alpha-1}(AL)^{1-\alpha}mK + (1-\alpha)K^{\alpha}A^{-\alpha}L^{1-\alpha}pA + (1-\alpha)K^{\alpha}A^{1-\alpha}L^{-\alpha}nL$
$= \alpha mK^{\alpha}(AL)^{1-\alpha} + (1-\alpha)pK^{\alpha}(AL)^{1-\alpha} + (1-\alpha)nK^{\alpha}(AL)^{1-\alpha} = [\alpha m + (\alpha-1)p + (\alpha-1)n]\,Y = [\alpha m + (\alpha-1)(p+n)]\,Y$.

Thus, $\frac{dY}{dt} = [\alpha m + (\alpha-1)(p+n)]\,Y(t)$.
So, aggregate output grows at a constant proportional rate $\alpha m + (\alpha-1)(p+n)$.

(c) For a specific period t, find the partial elasticity of the aggregate output with respect to capital and give an interpretation. (6 pts)

The partial elasticity of aggregate output with respect to capital is:
$\varepsilon_{YK} = \frac{\partial Y}{\partial K}\frac{K}{Y} = \alpha K^{\alpha-1}(AL)^{1-\alpha}\frac{K}{K^{\alpha}(AL)^{1-\alpha}} = \alpha K^{\alpha}(AL)^{1-\alpha}\frac{1}{K^{\alpha}(AL)^{1-\alpha}} = \alpha$.
The partial elasticity of the aggregate output with respect to capital is given by α. It shows by how much, in percentage terms, Y changes if K is, say, increased by 1% keeping L and A constant.

(d) For a specific period t, show that there is a diminishing productivity of capital and labour and that these two inputs are complementary. Give an interpretation of the derived results. (6 pts)

The marginal product of capital is:
$\frac{\partial Y}{\partial K} = MP_K = \alpha K^{\alpha-1}(AL)^{1-\alpha}$.
Marginal productivity of capital is decreasing:
$\frac{\partial MP_K}{\partial K} = \alpha(\alpha-1)K^{\alpha-2}(AL)^{1-\alpha} < 0$.
The marginal product of labour is:
$\frac{\partial Y}{\partial L} = MP_L = (1-\alpha)K^{\alpha}A^{1-\alpha}L^{-\alpha}$.
Marginal productivity of labour is decreasing:
$\frac{\partial MP_L}{\partial L} = -\alpha(1-\alpha)K^{\alpha}A^{1-\alpha}L^{-\alpha-1} < 0$.
Since $\frac{\partial MP_K}{\partial K} < 0$ and $\frac{\partial MP_L}{\partial L} < 0$, there is a diminishing productivity of capital and labour. Thus, using additional inputs will increase output, but adding more inputs result in a smaller increase in the output level.

Using cross-partial derivatives, we have:
$\frac{\partial MP_K}{\partial L} = \alpha(1-\alpha)K^{\alpha-1}A^{1-\alpha}L^{-\alpha} > 0$,
$\frac{\partial MP_L}{\partial K} = \alpha(1-\alpha)K^{\alpha-1}A^{1-\alpha}L^{-\alpha} > 0$.
Since $\frac{\partial MP_K}{\partial L} = \frac{\partial MP_L}{\partial K} > 0$, capital and labour are complementary inputs. The marginal productivity of one input is increased by having more of the other input available.

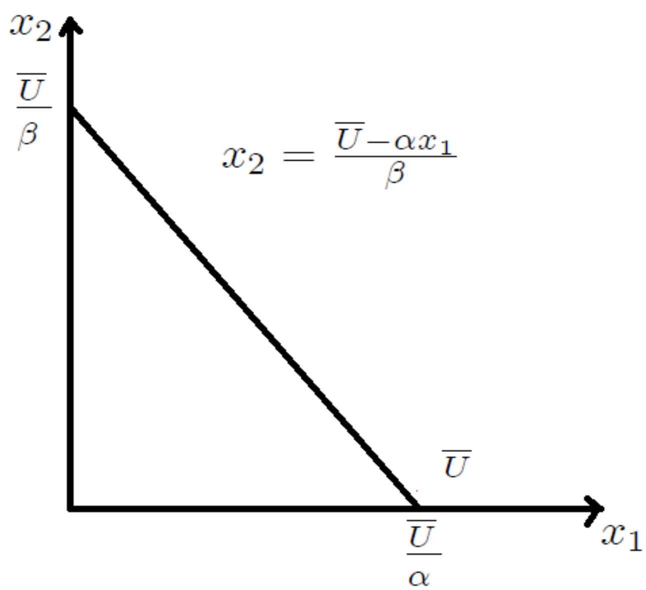
4. (28 pts) Consider a utility function $U(x_1, x_2) = \alpha x_1 + \beta x_2$. We know that U will be constant along a given indifference curve, i.e. $\overline{U}$ constant.

(a) Find the total differential for the utility function $U(x_1, x_2) = \alpha x_1 + \beta x_2$ and give an interpretation. (6 pts)

The total differential for the utility function is,
$dU = \frac{\partial U}{\partial x_1} dx_1 + \frac{\partial U}{\partial x_2} dx_2 \Rightarrow dU = \alpha dx_1 + \beta dx_2$.
The change in U (dU) is partially due to a change in commidity x_1 (dx_1) plus partially due to a change in commodity x_2 (dx_2).

(b) Draw a representative level curve for $U(x_1, x_2) = \overline{U}$. What is the slope of the curve? (6 pts)

From $U(x_1, x_2) = \overline{U} \Rightarrow \alpha x_1 + \beta x_2 = \overline{U}$ which represents a straight line. Thus, solving for x_2 we have, $x_2 = \frac{\overline{U} - \alpha x_1}{\beta}$. We have to draw the line $x_2 = \frac{\overline{U} - \alpha x_1}{\beta}$.
The slope of the curve is $-\frac{\alpha}{\beta}$.



(c) Use the expression for the total differential to illustrate that for a given indifference curve $\overline{U}$, $MRS = \alpha/\beta$. (6 pts)

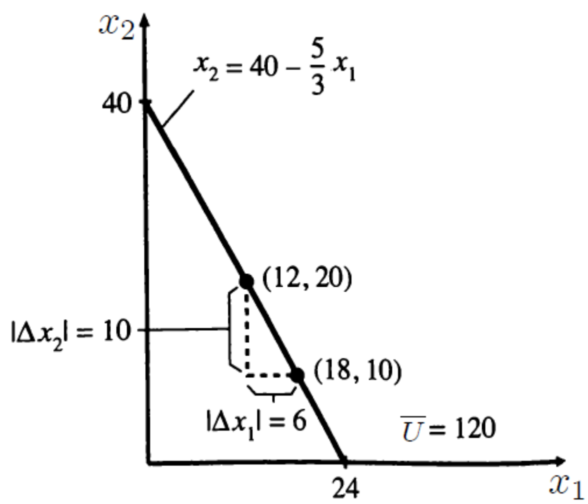
For a given indifference curve $\overline{U}$, $d\overline{U} = 0 \Rightarrow \frac{\partial U}{\partial x_1} dx_1 + \frac{\partial U}{\partial x_2} dx_2 = 0 \Rightarrow \alpha dx_1 + \beta dx_2 = 0 \Rightarrow -\frac{dx_2}{dx_1} = \frac{\alpha}{\beta} \Rightarrow MRS = \frac{\alpha}{\beta}$.

(d) Consider now that $\alpha = 5$, $\beta = 3$ and $\overline{U} = 120$. Draw the level curve for $\overline{U} = 120$ and show that $MRS = 5/3$. (5 pts)

We have, $x_2 = \frac{\overline{U} - \alpha x_1}{\beta}$. Now, we have to draw the line $x_2 = \frac{120 - 5x_1}{3} \Rightarrow x_2 = 40 - \frac{5}{3} x_1$.
The MRS in this case is, $MRS = \frac{\alpha}{\beta} = \frac{5}{3} \Rightarrow MRS = \frac{5}{3}$.

(e) Use the pair of points $(12, 20)$ and $(18, 10)$ to illustrate that the $MRS = 5/3$ which is what you derived from the total differential. (5 pts)

We know that $MRS = -\frac{dx_2}{dx_1} = -\frac{\Delta x_2}{\Delta x_1} = -\frac{(-10)}{8} = \frac{5}{3} \Rightarrow MRS = \frac{5}{3}$.
Note that since $U(x_1, x_2) = 5x_1 + 3x_2$ is a linear function, we do not need to take the limit as $\Delta x_1 \to 0$.



x_2
$x_2 = 40 - \frac{5}{3} x_1$
40
(12, 20)
$|\Delta x_2| = 10$
(18, 10)
$|\Delta x_1| = 6$
$\overline{U} = 120$
24
x_1