Solution.

CONCORDIA UNIVERSITY
Department of Economics

ECON 222 SECTIONS A, B
STATISTICAL METHODS II
Fall 2018 – MIDTERM 1
Sunday, October 14th, 14:30pm – 16:30pm

1. **(10 marks)** Differentiate the following functions with respect to x.

a. **(2 marks)** $f(x) = 5x^3$

$$f'(x) = 15x^2$$

b. **(2 marks)** $f(x) = \ln(x^4 + 3x^2)$

$$f'(x) = \frac{4x^3 + 6x}{x^4 + 3x^2}$$

c. **(2 marks)** $f(x) = e^{x^3}$

$$f'(x) = e^{x^3}(3x^2) = 3x^2 e^{x^3}$$

d. **(2 marks)** $f(x) = x^2 \ln(x^2)$

$$f'(x) = 2x \ln x^2 + \frac{2x}{x^2}(x^2) = 2x \ln x^2 + 2x$$

$$\Rightarrow f'(x) = 2x[\ln(x^2) + 1]$$

e. **(2 marks)** $f(x) = e^x/x$

$$f'(x) = \frac{xe^x - e^x}{x^2} \Rightarrow f'(x) = \frac{e^x(x-1)}{x^2}$$

2. **(6 marks)** Consider the following functions:

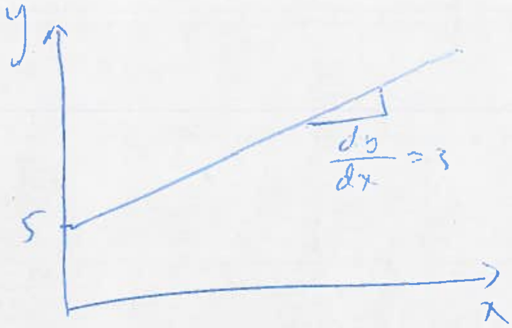
i. $y = 5 + 3x \quad x \geq 0,$
ii. $y = x^2 + 10x \quad x \geq 0.$

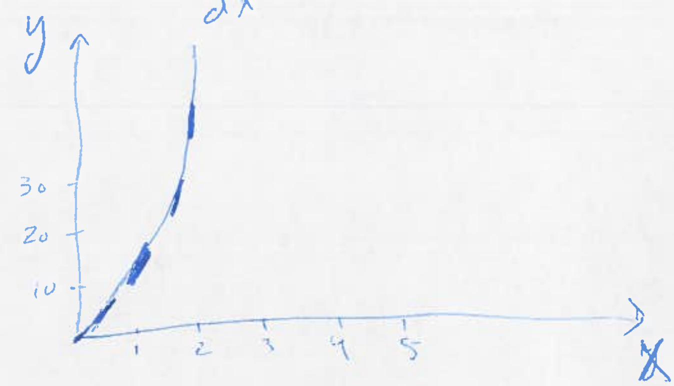
a. **(2 marks)** What are the slopes of the functions? Plot each function in a separate graph and indicate the slope on your graphs.

i) $\frac{dy}{dx} = 3$

ii) $y = x^2 + 10x$

$\frac{dy}{dx} = 2x + 10$





b. **(2 marks)** Compare the reaction of (y) to a one unit increase in (x) in both cases.

In (i), the change in (y) as a response to a change in (x) is constant and it is equal to 3. In the second case, however, the response of y is a function of x;
$\frac{dy}{dx} = 2x + 10 \Rightarrow$ it depends on the value of x.

c. **(2 marks)** For each function, find the value of (x) that would minimize (y).

i) At $x = 0$ $y = 5$ (This is a minimum)

ii) $\frac{dy}{dx} = 2x + 10 = 0 \Rightarrow \boxed{x = -5}, \boxed{y = -25}$
is the minimum. Since $x \geq 0 \Rightarrow$ at $x = 0$
y would be at the minimum. $(x = 0, y = 0)$

3. **(8 marks)** Consider the function f(x)= c where $1 \le x \le 10$.

a. **(2 marks)** Find the value of (c) that would make f(x) a proper probability density function.

Recall, $\int_1^{10} c\, dx = 1$

$\Rightarrow c x \Big|_1^{10} = 1 \quad \Rightarrow [10c - c] = 1$

$\Rightarrow \boxed{c = \frac{1}{9}}$

b. **(2 marks)** Compute E(x).

$E(x) = \frac{1}{9}\int_1^{10} x\, dx = \frac{1}{9}\left[\frac{x^2}{2}\right]_1^{10}$

$= \frac{1}{9}\left[\frac{100}{2} - \frac{1}{2}\right] = \frac{99}{18}$

c. **(2 marks)** Compute Var(x).

$E(x^2) = \frac{1}{9}\int_1^{10} x^2\, dx = \frac{1}{9}\left[\frac{x^3}{3}\right]_1^{10} = \frac{999}{27}$

$\Rightarrow Var(x) = E(x^2) - [E(x)]^2 = \left(\frac{999}{27}\right) - \left(\frac{99}{18}\right)^2 = 6.75$

d. **(2 marks)** Find the cumulative distribution function of x, F(x).

$F(x) = \frac{1}{9}\int_1^{x} dx = \frac{1}{9}[x]_1^x = \frac{1}{9}[x-1]$

$1 \le x \le 10$

4. **(4 marks)** Let X be a continuous random variable with a probability density function $f(x) = 3x^2/8$ where $0 \le x \le 2$.

   a. **(2 marks)** Compute $P(3/2 \le x \le 2)$

$$P\left(\frac{3}{2} \le x \le 2\right) = \frac{3}{8}\int_{3/2}^{2} x^2\,dx = \frac{3}{8}\left[\frac{x^3}{3}\right]_{3/2}^{2}$$

$$= \frac{3}{8}\left[\frac{8}{3} - \frac{3.375}{3}\right]$$

$$= 0.578$$

   b. **(2 marks)** Compute $P(x \le 1/4)$

$$P\left(x \le \frac{1}{4}\right) = \frac{3}{8}\int_{0}^{1/4} x^2\,dx = \frac{3}{8}\left[\frac{x^3}{3}\right]_{0}^{0.25}$$

$$= \frac{3}{8}\left[0.005208\right] = 0.001953$$

5. **(8 marks)** Suppose that X and Y are two random variables with joint density $f(x,y) = 6x^2y$ where $0 \le x \le 1$ and $0 \le y \le 1$,

   a. **(4 marks)** Briefly explain whether X and Y are independent.

$$f_x(x) = 6x^2\int_0^1 y\,dy = 6x^2\left[\frac{y^2}{2}\right]_0^1 = 6x^2\left(\frac{1}{2}\right) = 3x^2$$

$$f_y(y) = 6y\int_0^1 x^2\,dx = 6y\left[\frac{x^3}{3}\right]_0^1 = 6y\left[\frac{1}{3}\right] = 2y$$

$$f_x(x)\,f_y(y) = 3x^2(2y) = 6x^2y = f(x,y)$$

$\Longrightarrow$ X & Y are independent.

b. **(4 marks)** Find the conditional mean and the conditional variance of Y given X=3/4.

$$f(y|x) = \frac{f(x,y)}{f_x(x)} = \frac{6x^2y}{3x^2} = 2y$$

Note that $f(y|x) = f_y(y)$; This is not surprising because x & y are independent.

$$\Rightarrow E(y|x=\tfrac{3}{4}) = E(y)$$

& $Var(y|x=\tfrac{3}{4}) = Var(y)$

$$\Rightarrow E(y) = 2\int_0^1 y^2\,dy = 2\left[\frac{y^3}{3}\right]_0^1 = \frac{2}{3} = E(y|x=\tfrac{3}{4})$$

$$E(y^2) = 2\int_0^1 y^3\,dy = 2\left[\frac{y^4}{4}\right]_0^1 = \frac{1}{2} = E(y^2|x=\tfrac{3}{4})$$

$$\Rightarrow Var(y) = \frac{1}{2} - \left(\frac{2}{3}\right)^2$$

$$= \frac{1}{2} - \frac{4}{9} = 0.0556 = Var(y|x=\tfrac{3}{4})$$

6. **(6 marks)** **Suppose that A_1, A_2, A_3, A_4 is a random sample from a N (μ, $\sigma 2$) population, compare between the following estimators, which one will you choose? Explain.**

i. $\acute{A} = (1/2)A_1 + (1/4)A_2 + (1/6)A_3 + (1/8)A_4$,
ii. $\tilde{A} = (1/4)A_1 + (1/4)A_2 + (1/6)A_3 + (1/3)A_4$,
iii. $\hat{A} = (1/4)A_1 + (1/4)A_2 + (1/4)A_3 + (1/4)A_4$.

(i)

$E(\acute{A}) = \frac{1}{2}E(A_1) + \frac{1}{4}E(A_2) + \frac{1}{6}E(A_3) + \frac{1}{8}E(A_4)$

$= \mu[\frac{1}{2} + \frac{1}{4} + \frac{1}{6} + \frac{1}{8})$

$= \mu(\frac{50}{48})$

$\Rightarrow E(\acute{A}) = (\frac{50}{48})\mu$

(ii)

~~E(A~~

$E(\tilde{A}) = \mu[\frac{1}{4} + \frac{1}{4} + \frac{1}{6} + \frac{1}{3}]$

$= \mu[\frac{12}{12}]$

$\Rightarrow E(\tilde{A}) = \mu$

(iii)

$E(\hat{A}) = \mu[\frac{1}{4} + \frac{1}{4} + \frac{1}{4} + \frac{1}{4}]$

$\Rightarrow E(\hat{A}) = \mu(\frac{4}{4})$

$\Rightarrow E(\hat{A}) = \mu$

Note that $\acute{A}$ is biased $\Rightarrow$ we will not consider it any further.

On the other hand, both $\tilde{A}$ & $\hat{A}$ are unbiased estimators.

For $\tilde{A}$, $Var(\tilde{A}) = \sigma^2[(\frac{1}{4})^2 + (\frac{1}{4})^2 + (\frac{1}{6})^2 + (\frac{1}{3})^2]$ given that A_1, A_2, A_3, A_4 are independent.

$\Rightarrow Var(\tilde{A}) = (0.2639)\sigma^2$

For $\hat{A}$, $Var(\hat{A}) = \sigma^2[(\frac{1}{4})^2 + (\frac{1}{4})^2 + (\frac{1}{4})^2 + (\frac{1}{4})^2]$ given that A_1, A_2, A_3, A_4 are independent

$\Rightarrow Var(\hat{A}) = 0.25\sigma^2$

$\because Var(\hat{A}) < Var(\tilde{A})$

$\Rightarrow \hat{A}$ is a better estimator.

7. **(8 marks)** An investor believes that the daily return on one particular stock is normally distributed with mean μ and variance σ^2 and she proceeded by collecting 1000 random observations. Let X be the stock return and $\bar{X} = 1/N \sum_1^N Xi$. Also note that $\sum_1^N Xi = 150$ and $\sum_1^N (Xi - \bar{X})^2 = 900$.

a. **(2 marks)** Estimate (μ) by an unbiased estimator.

The sample average is an unbiased estimator

$$\Rightarrow \bar{X} = \frac{\sum X_i}{N} = \frac{150}{1000} \Rightarrow \boxed{\bar{X} = 0.15}$$

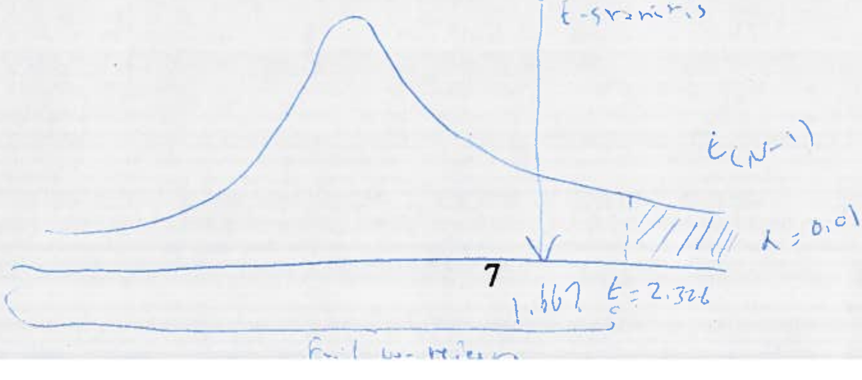
b. **(2 marks)** Estimate the population variance (σ^2) by an unbiased estimator.

$\hat{\sigma}^2 = \frac{\sum (X_i - \bar{X})^2}{N-1}$ is an unbiased estimator of σ^2

$$\Rightarrow \hat{\sigma}^2 = \frac{900}{999} = 0.9$$

c. **(2 marks)** Test the following with 99% confidence, explain your choice of the constructed statistic and demonstrate your answer graphically.
$H_0 : \mu = 0.10$
$H_1 : \mu > 0.10$

- $\alpha = 0.01 \Rightarrow t_c^{(0.01)} : P(t > t_c) = 0.01$ "Right tail test" ∵ H_1 is ">"
- $\Rightarrow t_c = 2.326$
- $t\text{-statistic} = \frac{\bar{X} - 0.1}{se(\bar{X})} = \frac{0.15 - 0.1}{\sqrt{\frac{0.9}{1000}}} = \frac{0.05}{0.03} = 1.667$
- ∵ $t\text{-statistic} < t_c \Rightarrow$ we fail to reject H_0



d. **(2 marks)** Transaction cost is a significant fixed cost incurred by the investor. Her profit, therefore, is given by the following function:

$$\pi = 200(1+\mu) - 50.$$

Test if the investor would make positive profit on average.

$E(\pi) = 150 + 200\, E(\mu)$

$\Rightarrow E(\pi) > 0$ if $E(\mu) > \frac{-150}{200} = -0.75$

$\Rightarrow H_0: \mu \leq -0.75$
$H: \mu > -0.75$

if we reject H_0, the data supports the alternative that the investor would make positive profit on average.

- for $\alpha = 0.01$, $t_c = +2.326$ "higher tail test"

- $t\text{-statistic} = \frac{0.15 - (-0.75)}{0.03} = 30$

- $\because$ t-statistic $> t_c \Rightarrow$ we reject H_0

$\Rightarrow$ The investor would make positive profit on average with 99% confidence.