Lecture 3

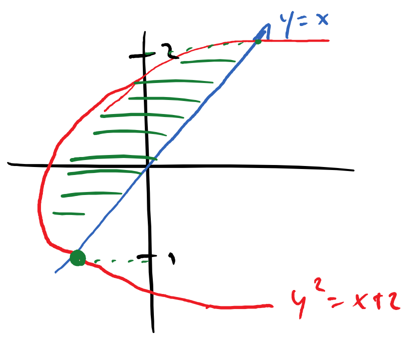
January 15, 2018 10:02 AM

Review:

Example 3: Area between $y^2 = x+2$, $y = x$

$$x = y^2 - 2$$
$$x = y$$

$$\Rightarrow y^2 - 2 = y \Rightarrow y = -1, 2$$



$$A = \int_{-1}^{2} (x_{right} - x_{left})\, dy$$

always right left

$$= \int_{-1}^{2} \left(y - (y^2 - 2)\right) dx$$

$$= \frac{y^2}{2} - \frac{y^3}{3} + 2y \Big|_{-1}^{2}$$

$$= 9/2$$

Calculating volume:

$$V = \lim_{n\to\infty} \sum_{i=1}^{n} A(x_i)\Delta x$$

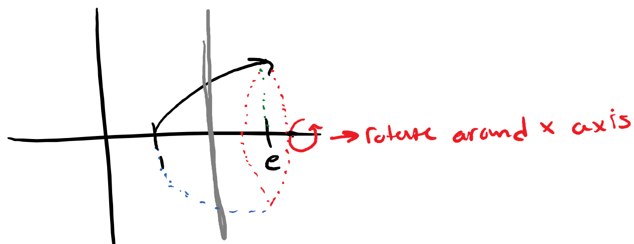
$$\Delta x = \frac{b-a}{n}$$

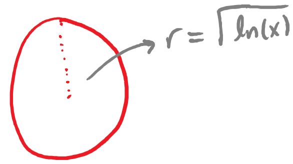
$$= \int_a^b A(x)\, dx$$

→ where $A(x)$ is the cross sectional area in a plane

Ex 5

$$f(x) = \sqrt{\ln(x)} \quad 1 \le x \le e$$





$$A(x) = \pi r^2 = \pi\left(\sqrt{\ln(x)}\right)^2 = \pi\left(\ln(x)\right)$$

$$V = \int_1^e A(x)\,dx = \int_1^e \pi \ln(x)\,dx$$

$$= \pi \int_1^e \ln(x)\,dx \quad \left| \begin{array}{ll} f = \ln(x) & g' = 1 \\ f' = & g = \end{array} \right.$$

$$= \pi \int_1^e x\ln(x) - x$$

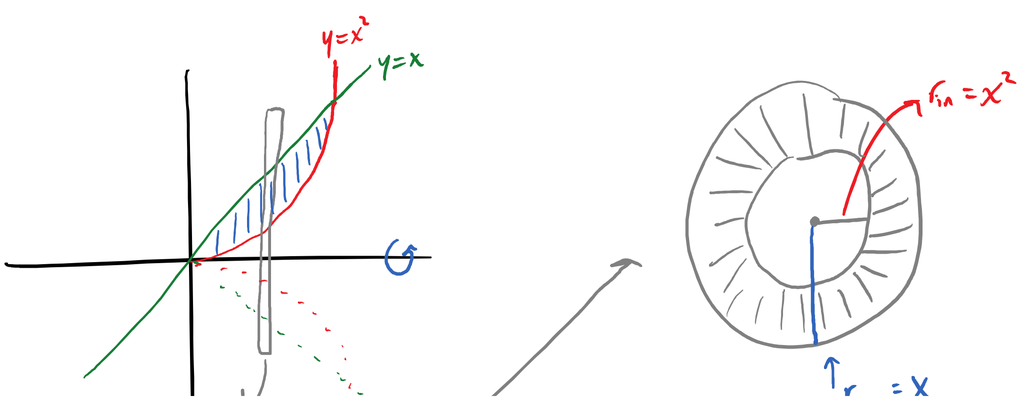
$$= \pi$$

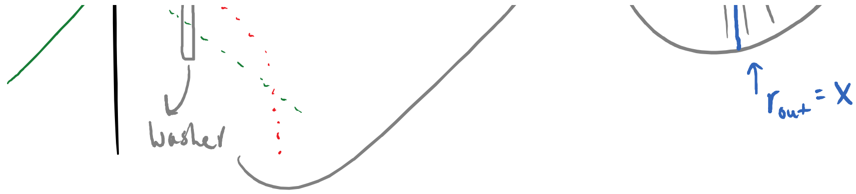
Ex 6:

$y = x$

$y = x^2$

* intersection points $x = x^2 \Rightarrow x = 0, 1$





$$A(x) = \pi r_{out}^2 - \pi r_{in}^2$$

$$= \pi x^2 - \pi (x^2)^2$$

$$= \pi x^2 - \pi x^4$$

$$V = \int_0^1 (\pi x^2 - \pi x^4)\, dx$$

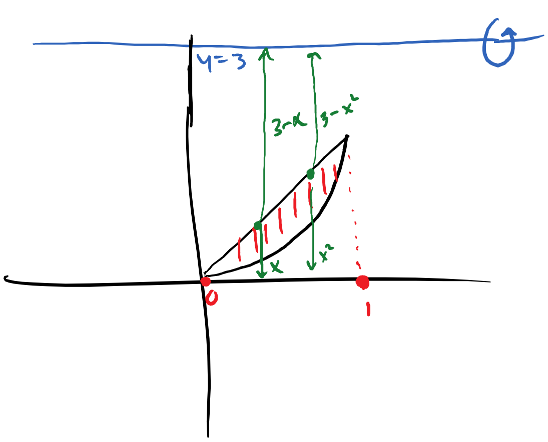
$$= \pi \int_0^1 (x^2 - x^4)\, dx$$

$$= \pi \int_0^1 \frac{x^3}{3} - \frac{x^5}{5}$$

$$= \left(\frac{\pi x^3}{3} - \frac{\pi x^5}{5}\right)\Big|_0^1$$

$$\boxed{= \frac{2\pi}{15}}$$

* if we rotate the region around $y=3$



$r_{out} = 3 - x^2$

$r_{in} = 3 - x$

$$A(x) = \pi r_{out}^2 - \pi r_{in}^2$$
$$= \pi(3-x^2)^2 - \pi(3-x)^2$$

$$V = \int_0^1 \left(\pi(3-x^2)^2 - \pi(3-x)^2\right) dx$$

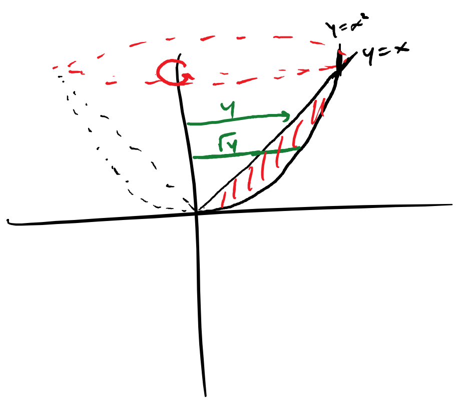
$$= \pi \int_0^1 \left(9 + x^4 - 6x^2 - 9 - x^2 - 6x\right) dx$$

$$= \pi \int_0^1 \left(x^4 - 7x^2 + 6x\right) dx$$

$$= \pi \left[\frac{x^5}{5} - \frac{7x^3}{3} + 3x^2\right]\Bigg|_0^1$$

$$\boxed{= \frac{13\pi}{15}}$$

What if we rotate the region around y-axis



$\Rightarrow$ $x = \sqrt{y}$
$x = y$

$r_{out} = \sqrt{y}$
$r_{in} = y$

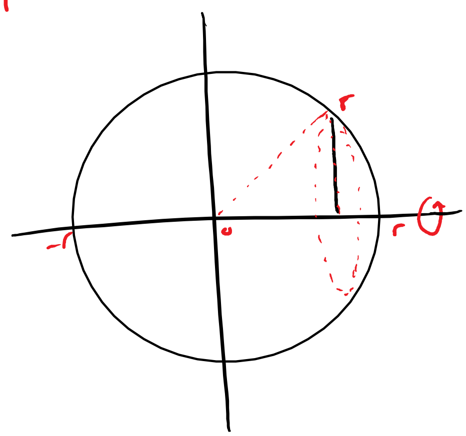
$$A = \pi r_{out}^2 - \pi r_{in}^2$$
$$= \pi(\sqrt{y})^2 - \pi y^2$$
$$= \pi y - \pi y^2$$

$$V = \int_0^1 A(y)\,dy = \int_0^1 (\pi y - \pi y^2)\,dy$$

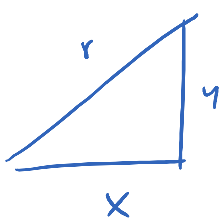
$$= \pi \int_0^1 (y - y^2)\,dy = \left[\frac{\pi y^2}{2} - \frac{\pi y^3}{3}\right]\Bigg|_0^1$$

$$\boxed{= \frac{\pi}{6}}$$

Ex 7



* The plane intersects the sphere is a circle with radius y



$$r^2 = y^2 + x^2 \Rightarrow y^2 = r^2 - x^2$$
$$y = \sqrt{r^2 - x^2}$$

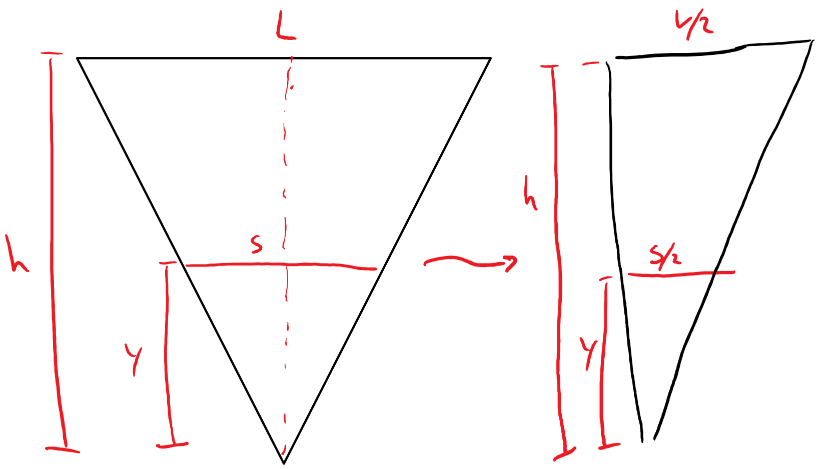
$$A = \pi y^2 = \pi\left(\sqrt{r^2 - x^2}\right)^2 = \pi(r^2 - x^2)$$

$$V = \int_{-r}^{r} A(x)\,dx = \int_{-r}^{r} \pi(r^2 - x^2)\,dx$$

$$= \pi\left(r^2 x - \frac{x^3}{3}\right)\Bigg|_{-r}^{r}$$

$$\boxed{= \frac{4}{3}\pi r^3}$$

Ex 8



$$\frac{y}{h} = \frac{\frac{s}{2}}{L/2}$$

$$s = \frac{Ly}{h}$$

$$A = s^2$$

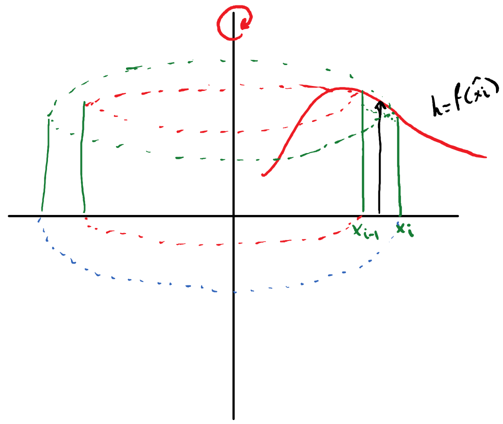
$$A(y) = \frac{L^2 y^2}{h^2}$$

$$V = \int_0^h A(y)\,dy = \int_0^h \frac{L^2 y^2}{h^2} \cdot dy$$

$$= \frac{L^2}{h^2}\int_0^h y^2\,dy = \frac{L^2}{3h^2} \cdot y^3 \Big|_0^h$$

$$\boxed{= \frac{L^2 h}{3}}$$

6.3 Volume by Cylindrical Shells



$X_i^* = \text{start}$

$\Delta x = x_i - x_{i-1}$

$X_i^* = \frac{x_i + x_{i-1}}{2}$

→ Two cylindrical shells

→ Inner one with $r_1 = x_{i-1}$

→ outer one with $r_2 = x_i$

$$V_i = V_2 - V_1 = \pi r_2^2 \cdot h - \pi r_1^2 \cdot h$$
$$= \pi (r_2^2 - r_1^2) h$$
$$= \pi (r_2 - r_1)(r_2 + r_1) h$$
$$= \pi (x_i - x_{i-1})(x_i + x_{i-1}) h$$
$$= \pi \cdot \Delta x \cdot 2\hat{x}_i \; f(\hat{x}_i) = 2\pi \hat{x}_i f(\hat{x}_i)\, dx$$

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Memorize this formula so you don't have to show proof.

$n \to \infty \quad \Delta x \to 0$

$$V = \int_a^b 2\pi x \, f(x) \, dx$$