Lecture 1

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Integration and its applications

Review:

$$F'(x) = f(x)$$

$$\int f(x)\,dx = F(x) + c$$

↳ where $F(x)$ is antiderivative of $f(x)$.

Ex 1 By substitution

$$\int_2^5 \frac{3x+6}{x^2+4x+1}\,dx$$

$$= 3\int_2^5 \frac{x+2}{x^2+4x+1}\,dx$$

$u = x^2 + 4x + 1$

$du = 2x + 4\,dx$

$= 2(x+2)\,dx$

$\frac{du}{2} = (x+2)\,dx$

$x = 2 \Rightarrow u = 4 + 8 + 1$

$= 13$

$x = 5 \Rightarrow u = 46$

$$= \frac{3}{2}\int_{13}^{46} \frac{du}{u}$$

$$= \frac{3}{2}\ln|u| \Big|_{13}^{46}$$

$$= \boxed{\frac{3}{2}\left(\ln(46) - \ln(13)\right)}$$

By parts:

$$= \int fg - \int f'g$$

Ex 2

$$\int x e^x \, dx \quad \Bigg| \begin{array}{ll} f = x & g' = e^x \\ f' = 1 & g = e^x \end{array}$$

$$= \int x e^x - \int e^x \, dx$$

$$\int e^x$$

$$= \boxed{x e^x - e^x + C}$$

Fundamental Theorem of Calculus

- if $F(x)$ is continuous on $[a,b]$ and $F(x)$ is any antiderivative of $f(x)$, then $\int_a^b f(x)\,dx = F(b) - F(a)$

7.8 Improper Integrals

$$\int_1^7 e^{-x}\,dx = F(7) - F(1)$$

$$\int_1^\infty e^{-x}\,dx = ?$$

* If $f(x)$ is continuous on $[a,b]$ but not at b or b might be infinity (∞) then,

$$\int_a^b f(x)\,dx = \lim_{d \to b} \int_a^d f(x)\,dx$$

↳ Improper integral

→ if the limit exists ⇒ The improper integral converges.

→ if the limit does not exist ⇒ the improper integral diverges.

* If $f(x)$ is continuous on $[a,b]$ but not at a or a is $(-\infty)$, then,

$$\int_a^b f(x)\,dx = \lim_{d\to a^+} \int_d^b f(x)\,dx$$

↳ is also an improper integral

Improper Integral type 1:

$$\int_a^b f(x)\,dx$$

→ a is $-\infty$ or/and b is ∞

* $$\int_1^\infty e^{-x}\,dx = \lim_{b\to\infty} \int_1^b e^{-x}\,dx$$

$$= \lim_{b\to\infty} \left[-e^{-x}\right]_1^b$$

$$= \lim_{b\to\infty} \left[-e^{-b} - (-e^{-1})\right]$$

$$= \lim_{b\to\infty} \left[-\underbrace{\cancel{e^{-b}}}_{0} + e^{-1}\right]$$

$$\boxed{= \frac{1}{e}}$$

Ex 2

$$\int_2^\infty \frac{1}{\sqrt{x}}\,dx = \lim_{b\to\infty} \int_2^b \frac{1}{\sqrt{x}}\,dx$$

$$= \lim_{b\to\infty} \left[\frac{x^{\frac{1}{2}}}{\frac{1}{2}} \Big|_2^b \right]$$

$$= \lim_{b\to\infty}\left[\frac{x^{1/2}}{1/2}\Big|_2\right]$$

$$= \lim_{b\to\infty}\left[2\sqrt{b} - 2\sqrt{2}\right]$$

$$\boxed{= \infty}$$

↳ Improper integral diverges (Limit does not exist)

Ex 3

$$\int_{-\infty}^{0} e^x\,dx = \lim_{b\to-\infty}\int_b^0 e^x\,dx$$

$$= \lim_{b\to-\infty}\left(e^x\Big|_b^0\right)$$

$$= \lim_{b\to-\infty}\left[1 - e^b\right]$$

$$\boxed{= 1}$$

↳ Convergent (Limit exists)

ex 4

$$\int_{-\infty}^{\infty} xe^{-x^2}\,dx$$ → split into 2 parts when $\left(\int_{-\infty}^{\infty}\right)$

$$= \int_{-\infty}^{0} xe^{-x^2}\,dx + \int_0^{\infty} xe^{-x^2}\,dx$$

$$= \lim_{a\to-\infty}\int_a^0 xe^{-x^2}\,dx + \lim_{b\to\infty}\int_0^b xe^{-x^2}\,dx$$

Substitution $$= \lim\left[-\frac{1}{2}e^{-x^2}\Big|^0\right] + \lim\left[-\frac{1}{2}e^{-x^2}\Big|^b\right]$$

Substitution

$u = x^2$

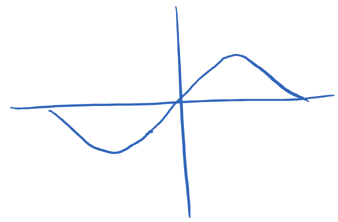
$du = -2x\,dx$

$$= \lim_{a\to-\infty}\left[-\frac{1}{2}e^{-x^2}\Big|_a^0\right] + \lim_{b\to\infty}\left[-\frac{1}{2}e^{-x^2}\Big|_0^b\right]$$

$$= \lim_{a\to-\infty}\left[-\frac{1}{2} + \frac{1}{2}\cancel{e^{-a^2}}_{\,0}\right] + \lim_{b\to\infty}\left[-\frac{1}{2}\cancel{e^{-b^2}}_{\,0} + \frac{1}{2}\right]$$

$$= -\frac{1}{2} + \frac{1}{2}$$

$$\boxed{=0} \rightarrow$$



Could look like this.

Improper Integral type 2:

→ when $f(x)$ has a discontinuity (typically a vertical asymptote) on $[a,b]$

Ex 4

$$\int_{-1}^{1} \frac{1}{x^2}\,dx$$

$$\int_{-1}^{1} \frac{1}{x^2}\,dx = \left(-\frac{1}{x}\right)\Big|_{-1}^{1} = -2$$

Wrong Solution

$f(x) = \frac{1}{x^2}$ $\quad D_f = \mathbb{R} - \{0\} \rightarrow$ Type 2

$$\int_{-1}^{1} \frac{1}{x^2}\,dx = \int_{-1}^{0} \frac{1}{x^2}\,dx + \int_{0}^{1} \frac{1}{x^2}\,dx$$

$$= \lim_{a\to0^-}\int_{-1}^{a} \frac{1}{x^2}\,dx + \lim_{b\to0^+}\int_{b}^{1} \frac{1}{x^2}\,dx$$

$$= \lim_{a\to0^-}\left[-\frac{1}{x}\Big|_{-1}^{a}\right] + \lim_{b\to0^+}\left[-\frac{1}{x}\Big|_{b}^{1}\right]$$

$$= \lim \left[\ \frac{1}{\ } - \left(-\frac{1}{\ }\right)\right] + \lim_{+}\left[-\frac{1}{1} - \left(-\frac{1}{b}\right)\right]$$

$$= \lim_{a\to 0^-}\underbrace{\left[-\frac{1}{a}-\left(-\frac{1}{-1}\right)\right]}_{\infty} + \lim_{b\to 0^+}\underbrace{\left[-\frac{1}{1}-\left(-\frac{1}{b}\right)\right]}_{\infty}$$

$= \infty \rightarrow$ diverges (Limit does not exist)

Ex 5

$$\int_0^4 \frac{1}{(x-2)^2}\,dx \qquad D_f = \mathbb{R} - \{2\}$$

$$= \int_0^{2} \frac{1}{(x-2)^2}\,dx + \int_{2}^4 \frac{1}{(x-2)^2}\,dx$$

$$= \lim_{a\to 2^-}\int_0^a \frac{1}{(x-2)^2}\,dx + \lim_{b\to 2^+}\int_b^4 \frac{1}{(x-2)^2}\,dx$$

$$= \lim_{a\to 2^-}\left[-\frac{1}{(x-2)}\Big|_0^a\right] + \lim_{b\to 2^+}\left[-\frac{1}{(x-2)}\Big|_b^4\right]$$

$$= \lim_{a\to 2^-}\left[-\frac{1}{(a-2)}+\frac{1}{-2}\right] + \lim_{b\to 2^+}\left[-\frac{1}{(4-2)}-\frac{1}{(b-2)}\right]$$

$\boxed{= \infty} \rightarrow$ diverges (Limit does not exist)