Math 184 Section 104 Assignment 5

Due Thursday, October 29, 2009.

1) (8 points) Find the intervals where the following functions are increasing and where they are decreasing. Also find all critical points, locations of local maxima and minima, global maxima and minima, inflection points, and regions of positive and negative concavity.

a) $f(x) = e^x(x^2 + 4x)$ (Try graphing this on a calculator if you have one to get an idea of why we want to be able to find critical points by hand!)

$$f'(x) = e^x(x^2 + 4x) + e^x(2x + 4) = e^x(x^2 + 6x + 4)$$
$$f''(x) = e^x(x^2 + 6x + 4) + e^x(2x + 6) = e^x(x^2 + 8x + 10)$$

Critical points: $f'(x) = 0$ *or* $(x^2 + 6x + 4) = 0$

$$x = \frac{-6 \pm \sqrt{36 - 16}}{2} = \frac{-6 \pm \sqrt{20}}{2} = -3 \pm \sqrt{5}$$

Increasing: $x \in (-\infty, -3 - \sqrt{5})$ *and* $x \in (-3 + \sqrt{5}, \infty)$
Decreasing: $x \in (-3 - \sqrt{5}, -3 + \sqrt{5})$.

Local Maximum: $f(-3 - \sqrt{5})$.
Local Minimum: $f(-3 + \sqrt{5})$.

Concavity: $f''(x) = 0$ *or* $(x^2 + 8x + 10) = 0$

$$x = \frac{-8 \pm \sqrt{64 - 40}}{2} = \frac{-6 \pm \sqrt{24}}{2} = -4 \pm \sqrt{6}$$

Concave up: $x \in (-\infty, -4 - \sqrt{6})$ *and* $x \in (-4 + \sqrt{6}, \infty)$
Concave down: $x \in (-4 - \sqrt{6}, -4 + \sqrt{6})$.

Global Maximum:

$$\lim_{x \to \infty} f(x) \to \infty$$

No global maximum.

Global Minimum:

$$\lim_{x \to -\infty} f(x) = 0$$

$$f(-3 + \sqrt{5}) = e^{-3+\sqrt{5}}((-3 + \sqrt{5})^2 + 4(-3 + \sqrt{5})) = e^{-3+\sqrt{5}}(2 - 2\sqrt{5}) < 0$$

Global minimum is $e^{-3+\sqrt{5}}(2 - 2\sqrt{5})$ *at* $x = -3 + \sqrt{5}$.

b) $f(x) = \frac{(3x-1)^2}{(x+4)^2}$

$$f'(x) = \frac{2(3)(3x-1)(x+4)^2 - 2(x+4)(3x-1)^2}{(x+4)^4}$$

$$f'(x) = \frac{2(3x-1)(x+4)[3(x+4)-(3x-1)]}{(x+4)^4} = \frac{26(3x-1)}{(x+4)^3}$$

$$f''(x) = \frac{26(3)(x+4)^3 - 3(x+4)^2(26)(3x-1)}{(x+4)^6}$$

$$f''(x) = \frac{26(3)(x+4)^2((x+4)-(3x-1))}{(x+4)^6} = \frac{78(5-2x)}{(x+4)^4}$$

Critical points: *$f'(x) = 0$ or $(3x-1) = 0$ and $f'(x)$ undefined.*

$$x = \frac{1}{3}, -4$$

Increasing: $x \in (-\infty, -4)$ and $x \in (1/3, \infty)$
Decreasing: $x \in (-4, 1/3)$.

Local Minimum: $f(1/3) = \frac{(3(1/3)-1)^2}{((1/3)+4)^2} = 0$.

Concavity: *$f''(x) = 0$ or $(5-2x) = 0$ and $f''(x)$ undefined.*

$$x = \frac{5}{2}, -4$$

Concave up: $x \in (-\infty, -4)$ and $x \in (5/2, \infty)$
Concave down: $x \in (-4, 5/2)$.

Global Maximum: There's an asymptote at $x = -4$

$$\lim_{x \to 4^-} f(x) \to \infty \qquad \lim_{x \to 4^+} f(x) \to \infty$$

No global maximum.

Global Minimum: Can see that $f(x) \geq 0$ for all x.

Global minimum is 0 at $x = 1/3$.

2 (2 points) Find the intervals where $f(x) = [\ln(x^2 - 3x)]^2$ is increasing and where they are decreasing. Also find all critical points, locations of local maxima and minima. State the domain of the function.

Domain of the function is when $(x^2 - 3x) > 0$. Or $x(x-3) > 0 \quad \Rightarrow \quad x \in \{(\infty, 0), (3, \infty)\}$

$$f'(x) = 2[\ln(x^2 - 3x)] \left(\frac{1}{(x^2 - 3x)} (2x - 3) \right) = \frac{2(2x-3)[\ln(x^2 - 3x)]}{(x^2 - 3x)}$$

On the domain, the denominator is always positive. $f'(x) = 0$ *when* $x = 3/2$*, but this point is not on the domain, or when* $\ln(x^2 - 3x) = 0$*. this gives* $x^2 - 3x = 1$

$$x = \frac{3 \pm \sqrt{9 - 4(-1)}}{2} = \frac{3 \pm \sqrt{13}}{2}$$

Increasing: $x \in \left[(\frac{3-\sqrt{13}}{2}, 0), (\frac{3+\sqrt{13}}{2}, \infty) \right]$.

Decreasing: $x \in \left[(-\infty, \frac{3-\sqrt{13}}{2}), (3, \frac{3+\sqrt{13}}{2}) \right]$.

To find global max/min, need to look at limits. ln *is an increasing function, so*

$$\lim_{x \to \infty} f(x) = \lim_{x \to \infty} [\ln(x^2 - 3x)]^2 \to \infty$$

And as $(x^2 - 3x) \to 0$, $\ln(x^2 - 3x) \to \infty$ *as well. So there is not global maximum.*

3) (4 points) Jane sleeps 8 hours per day. She has a summer job that allows her to work any number of hours per day (up to 8 hours) at \$6 per hour. The remaining hours are spent doing mathematics. Her mother gives her an allowance of \$6 per day. The amount of daily happiness that Jane gets from getting y dollars and spending h hours doing mathematics is given by $U(y, h)$, where

$$U(y, h) = y^{1/3} h^{2/3}.$$

How many hours per day should she work to maximize her happiness?.
Do this problem in two steps:
a) Let x be the number of hours she works. Express y in terms of x. Consider that there are 24 hours in a day, she sleeps for 8 and the rest of the hours are either working or doing mathematics. So express h in terms of x.
Then find U as a function of x.

The number of hours she has left for working and math is $24 - 8 = 16$*. With* x *being the number of hours she works,* $x + h = 16$ *or:*

$$h = 16 - x$$

The amount of money she gets comes from working and allowance. So,

$$y = 6x + 6$$

Thus, plug these into the equation for U*:*

$$U(x) = (6x + 6)^{1/3} (16 - x)^{2/3}$$

b) Optimize $U(x)$ from part (a).

Now find the maximum of with respect to x. The calculations might get a little messy, but straight forward. Use product and chain rules.

$$U'(x) = \frac{1}{3}(6x+6)^{-2/3}(6)(16-x)^{2/3} + \frac{2}{3}(16-x)^{-1/3}(-1)(6x+6)^{1/3}$$

$$U'(x) = 2(6x+6)^{-2/3}(16-x)^{2/3} - \frac{2}{3}(16-x)^{-1/3}(6x+6)^{1/3}$$

$$U'(x) = 2(6x+6)^{-2/3}(16-x)^{-1/3}\left[(16-x) - \frac{1}{3}(6x+6)\right]$$

$$U'(x) = \frac{2(16-3x-2)}{(6x+6)^{2/3}(16-x)^{1/3}}$$

So $U'(x) = 0$ when $14 - 3x = 0$ or $x = 14/3$.
She should work for about 4.6 hours.

4) (4 points) Suppose that when a busy restaurant charges \$7 for its tomato appetizer, an average of 60 people order the dish each night. When it drops the price of the appetizer to \$5, the number ordering it rises to 66. Assume that the demand q is a linear function of the price p. If each appetizer costs the restaurant \$3 to make, use calculus to find the price it should charge to maximize its profit from the appetizer.

Find the linear relation between q and p

$$q = \frac{66-60}{5-7}(p-7) + 60$$
$$q = -3p + 81$$

Find the profit as a function of price

$$C(p) = 3q = 3(-3p+81) = -9p + 243$$
$$R(p) = p \cdot q = p(-3p+81) = -3p^2 + 81p$$
$$P(p) = R(p) - C(p) = -3p^2 + 81p - (-9p + 243) = -3p^2 + 90p - 243$$

Optimize the profit:

$$P'(p) = -6p + 90 \quad \Rightarrow \quad p = 15$$

Verify that is is a maximum, $P''(p) = -6 < 0$. So the restaurant should charge \$ 15 for the appetizer.

5) (4 points) A carpenter has been asked to build an open box with a square base, where an open box means a box without a top. The sides of the box will cost \$2 per square meter, and the base will cost \$5 per square meter. What are the dimensions of the box of maximal volume that can be constructed for \$60 ?

Find the equation of the cost (which is fixed). Let x be the side length of the base and h, the height. The area of the base is x^2 and the area of the 4 sides is $4xh$. The cost is:

$$60 = 5 \cdot x^2 + 2 \cdot (4xh)$$
$$60 = 5x^2 + 8xh$$

The volume of a rectangular prism is $V = l \cdot w \cdot h$. So the volume that needs to maximized is:

$$V = x^2 h$$

It is easier to find $V(x)$ in this case. So, express $h(x) = \frac{15}{2x} - \frac{5x}{8}$.

$$V(x) = x^2 \left(\frac{15}{2x} - \frac{5x}{8} \right) = \frac{15x}{2} - \frac{5x^3}{8}$$

Optimize,

$$\frac{dV}{dx} = \frac{15}{2} - \frac{15x^2}{8} \quad \Rightarrow \quad x^2 = 4$$

Thus, $x = 2$, and $h = \frac{15}{4} - \frac{10}{8} = \frac{5}{2}$.
Verify that this is optimal,

$$\frac{d^2}{dx^2} V(2) = -\frac{30(2)}{8} < 0$$

The optimal dimensions are $x = 2$ and $h = 5/2$ with maximum volume of $V = 10$

6) (4 points) The University bookstore sells 8000 copies of the calculus book per year. In costs \$40 to process each new order and the carrying cost is \$2 per book, to be figured on the maximum inventory during an order-reorder period. How many times a year should orders be placed?

Let x be the size of shipment and r be the number of orders.

$$C = 40r + 2(x/2) = 40r + x$$
$$xr = 8000$$

Express as a function of r, since asked for number of orders: $C(r) = 40r + \frac{8000}{r}$. Optimize,

$$C'(r) = 40 - \frac{8000}{r^2} \quad \Rightarrow \quad r^2 = \frac{8000}{40}$$

$$r = \sqrt{200} = 10\sqrt{2}$$

So, orders should be placed about 14 times. (Although the closest number of orders so that both x and r are integers is 16).

7) (2 points) Find all values of a such that the polynomial $f(x) = x^3 - 3x^2 + ax + 1$ has a local maximum.

Maximum requires for derivative of $f(x)$ to be 0 and for the second derivative to be negative. So,

$$f'(x) = 3x^2 - 6x + a \quad f''(x) = 6x - 6$$

Maximum can only happen when $6x - 6 < 0$ or $x < 1$. Solve for the critical points from f':

$$x = \frac{6 \pm \sqrt{36 - 4(3)(a)}}{2 \cdot 3} = \frac{6 \pm \sqrt{12(3-a)}}{6} = 1 \pm \frac{\sqrt{12}}{6}\sqrt{(3-a)}$$

Clearly to have a maximum there ***must*** *be a critical point less than 1. This will happen whenever the term $\frac{\sqrt{12}}{6}\sqrt{(3-a)}$ is real. That is, $a < 3$. Then there will be 2 critical points (one greater and one less than 1).*

Bonus: Use logarithmic differentiation (take logarithms of both sides) to find $f'(5)$ where $f(x) = g(x)h(x)u(x)$ and $g'(5)/g(5) = h'(5)/h(5) = u'(5)/u(5) = 1$ and $f(5) = 2$.

$$\ln f(x) = \ln[g(x)h(x)u(x)] = \ln g(x) + \ln h(x) + \ln u(x)$$

Differentiate:

$$\frac{f'(x)}{f(x)} = \frac{g'(x)}{g(x)} + \frac{h'(x)}{h(x)} + \frac{u'(x)}{u(x)}$$

Plug in the given values:

$$\frac{f'(5)}{f(5)} = \frac{g'(5)}{g(5)} + \frac{h'(5)}{h(5)} + \frac{u'(5)}{u(5)} = 1 + 1 + 1 = 3$$

$$\frac{f'(5)}{f(5)} = 3 \quad \Rightarrow \quad f'(5) = 3 \cdot 2 = 6$$