Question 1- Calculate:

$$\lim_{x \to 2} \frac{x^2 - x - 2}{x - 2}$$

A) 1 **B)** 2 **C)** 3 **D)** 4 **E)** 5

$$\frac{x^2 - x - 2}{x - 2} = \frac{(x-2)(x+1)}{x-2} \quad \text{So} \quad \lim_{x \to 2} x+1 = 3$$

Question 2- For what value of x does $f(x) = x\ln(x)$ have slope equal to 0?

A) e **B)** e^{-1} **C)** -1 **D)** 0 **E)** None.

$f'(x) = x \cdot \frac{1}{x} + \ln(x) = 1 + \ln(x)$

Set $1 + \ln(x) = 0$

$\ln(x) = -1$

$x = e^{-1}$

Question 3-Use implicit differentiation to find $\frac{dy}{dx}$ at $(1, 1)$ when

$$x^3 + 2x^2y - y^3 + 3y - 5 = 0$$

A) $\frac{5}{3}$ **B)** $\frac{3}{2}$ **C)** $-\frac{1}{2}$ **D)** $\frac{2}{5}$ **E)** $-\frac{7}{2}$

$$3x^2 + 4xy + 2x^2\frac{dy}{dx} - 3y^2\frac{dy}{dx} + 3\frac{dy}{dx} = 0$$

Plug in $(1,1)$

$$3 + 4 + 2\frac{dy}{dx} - 3\frac{dy}{dx} + 3\frac{dy}{dx} = 0$$

$$7 + 2\frac{dy}{dx} = 0$$

$$\frac{dy}{dx} = \frac{-7}{2}$$

Question 4- Consider the function $f(x) = x^3 - 3x^2 + 1$. Which of the following statements is correct?

A) There is an inflection point at $x = 1$. **B)** There is a local max at $x = 1$.
C) There is a local min at $x = 1$. **D)** There is a local max at $x = 4$.
E) There is a local max at $x = 4$.

$$f'(x) = 3x^2 - 6x = 3x(x-2)$$

So CP's are at $0, 2$.

$f''(x) = 6x - 6$. Possible IP at $x = 1$.

	$x=1$	
$f(x)''$	$-$	$+$
$f(x)$	$\cap$	$\cup$

So there is an IP at $x=1$

Question 5- Consider the function $g(x) = xe^{-4x}$. Over what interval is the function increasing?

A) $(-\infty, \frac{1}{4})$ **B)** $(\frac{1}{4}, \infty)$ **C)** $(0, 4)$ **D)** $(1, \frac{5}{4})$ **E)** $(-1, \frac{5}{4})$

$g'(x) = e^{-4x} - 4xe^{-4x} = (1-4x)e^{-4x}$ CP at $x = \frac{1}{4}$

	$x = 1/4$	
$f'(x)$	$+$	$-$
$f(x)$	$\nearrow$	$\searrow$

Question 6- Evaluate:

$$\int_0^\infty \frac{x}{(1+x^2)^2} \mathrm{dx}$$

A) 1 **B)** $\frac{1}{2}$ **C)** 2 **D)** $2\ln(2)$ **E)** divergent

$= \lim_{b \to \infty} \int_0^b \frac{x}{(1+x^2)^2}\,dx$

$\int_0^b \frac{x}{(1+x^2)^2}\,dx$ $\boxed{u = 1+x^2 \quad du = 2x\,dx}$

$= \frac{1}{2}\int \frac{2x\,dx}{(1+x^2)^2} = \frac{1}{2}\int \frac{du}{u^2} = \frac{1}{2}\frac{u^{-1}}{-1} = \frac{-1}{2(1+x^2)}$

So $\int_0^b \frac{x}{(1+x^2)^2}\,dx = \left.\frac{-1}{2(1+x^2)}\right|_0^b = \frac{-1}{2(1+b^2)} - \left(\frac{-1}{2}\right)$

Back to limit $= \lim_{b \to \infty} \left(\frac{-1}{2(1+b^2)} + \frac{1}{2}\right) = 1/2$

Question 7- Suppose $f'(x) = 4x^3 - 3x^2 + 4x - 3$ and $f(0) = -6$. Find $f(2)$.

A) 0 B) 1 C) 2 D) 3 E) 4

$f(x) = \int (4x^3 - 3x^2 + 4x - 3)\,dx = x^4 - x^3 + 2x^2 - 3x + C$

$f(0) = C = -6$. So $f(x) = x^4 - x^3 + 2x^2 - 3x - 6$

$f(2) = 16 - 8 + 8 - 6 - 6 = 4$

Question 8- Suppose that for a certain product, the demand function is given by $D(x) = 20 - 3x^2$ and the supply function is given by $S(x) = x^2 + 4$. Calculate the consumer surplus.

A) 24 B) 16 C) 28 D) 27 E) 13

Intersection

$20 - 3x^2 = x^2 + 4$

$16 = 4x^2$

$4 = x^2$

$x = \pm 2$

Equilibrium at (2,8)

So $CS = \int_0^2 [(20 - 3x^2) - 8]\,dx$

$= \int_0^2 (12 - 3x^2)\,dx = 12x - x^3 \Big|_0^2$

$= 24 - 8 = 16$

Question 9- Calculate:

$$\int_1^e x\ln(x)dx$$

A) e^2 **B)** $\frac{e^2}{2}$ **C)** $\frac{e-1}{2}$ **D)** $\frac{e^2+1}{4}$ **E)** $\frac{e+3}{2}$

$$\int x \ln(x)\, dx$$

$$\boxed{\begin{array}{ll} u = \ln(x) & v = \frac{x^2}{2} \\ du = \frac{1}{x}dx & dv = x\,dx \end{array}}$$

$$= \frac{x^2}{2}\ln(x) - \int \frac{1}{x}\cdot\frac{x^2}{2}dx$$

$$= \frac{x^2}{2}\ln(x) - \frac{x^2}{4}$$

$$\frac{x^2}{2}\ln(x) - \frac{x^2}{4}\Big|_1^e$$

$$= \frac{e^2}{2} - \frac{e^2}{4} - \left(0 - \frac{1}{4}\right)$$

$$= \frac{e^2}{4} + \frac{1}{4} = \frac{e^2+1}{4}$$

Question 10- If $f(x,y) = e^{x^2+y^2}$, what is $f_{xy}(1,1)$?

A) e^2 **B)** $2e^2$ **C)** $3e^2$ **D)** $4e^2$ **E)** $5e^2$

$$f_x = \left(e^{x^2+y^2}\right)2x$$

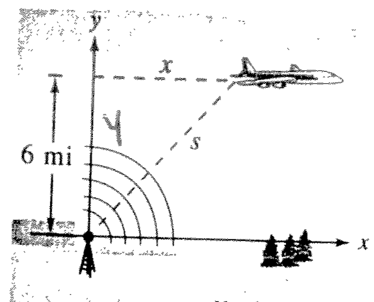
$$f_{xy} = \left(e^{x^2+y^2}\right)(2x)(2y)$$

$$= 4xy\, e^{x^2+y^2}$$

$$f_{xy}(1,1) = 4e^2$$

Long Answer Question 1 (10 points)

An airplane flying at a height of 6 miles passes directly over a radar antenna. (See picture.) When the airplane is 10 miles away from the antenna (so $s = 10$), the radar detects that the distance between the plane and the radar is changing at a rate of 240 miles per hour. What is the speed of the airplane? **HINT: This question requires related rates.**



Not drawn to scale

If $y=6$ and $s=10$, then $x=8$

We have $x^2+y^2=s^2$

$$2x\frac{dx}{dt} + 2y\frac{dy}{dt} = 2s\frac{ds}{dt}$$

We know $\frac{dy}{dt}=0$ & $\frac{ds}{dt}=240$

So $$2(8)\frac{dx}{dt} + 2(6)\cdot 0 = 2(10)(240)$$

$$16\frac{dx}{dt} = 4800$$

So $\frac{dx}{dt} = 300$

Long Answer Question 2 (8 points)

A manufacturer can produce **Red Sox hats** at a cost of 9 dollars each. They have been selling the hats at a price of 21 dollars each. At this price they sell 5400 hats per month. For each 1 dollar reduction in price, they sell 300 more hats.

(a) (4 points) Find the demand function. You may assume it is linear.

(b) (4 points) Find the profit function.

x = # of hats

p	x
21	5400
20	5700

$$m = \frac{\Delta p}{\Delta x} = \frac{21-20}{5400-5700} = \frac{-1}{300}$$

$p = mx + b = \frac{-1}{300}x + b$. Plug in ~~(21, 5400)~~ (5400, 21)

$$21 = \frac{-1}{300}(5400) + b$$

$21 = -18 + b \quad b = 39$. So $p = \frac{-1}{300}x + 39$

$$\text{Revenue} = \frac{-1}{300}x^2 + 39x$$

$$\text{Cost} = 9x$$

So $\text{Profit} = -\frac{1}{300}x^2 + 30x$

Long Answer Question 3 (6 points)

Find the absolute maximum for the function $f(x) = x^3 - 3x^2 + 3x + 2$ on the interval $[0, 2]$. Be sure to explain your answer.

$f'(x) = 3x^2 - 6x + 3 = 3(x^2 - 2x + 1) = 3(x-1)^2$

Use extreme value theorem:

x	$f(x)$
0	2
1	3
2	4

So max is at $(2, 4)$

Long Answer Question 4 (14 points)
Consider the two functions:
$f(x) = x - 1$ and $f(x) = x^2 + 2x - 3$ $= (x+3)(x-1)$

(a) **(2 points)** Find the intersection points of the graphs of the two functions.

(b) **(6 points)** On the next page, graph these functions, and shade the region between the graphs of f and g for x such that $-1 \leq x \leq 2$.

(c) **(6 points)** Find the area of the shaded region.

a) $x - 1 = x^2 + 2x - 3$

$x^2 + x - 2 = 0$

$(x+2)(x-1)$

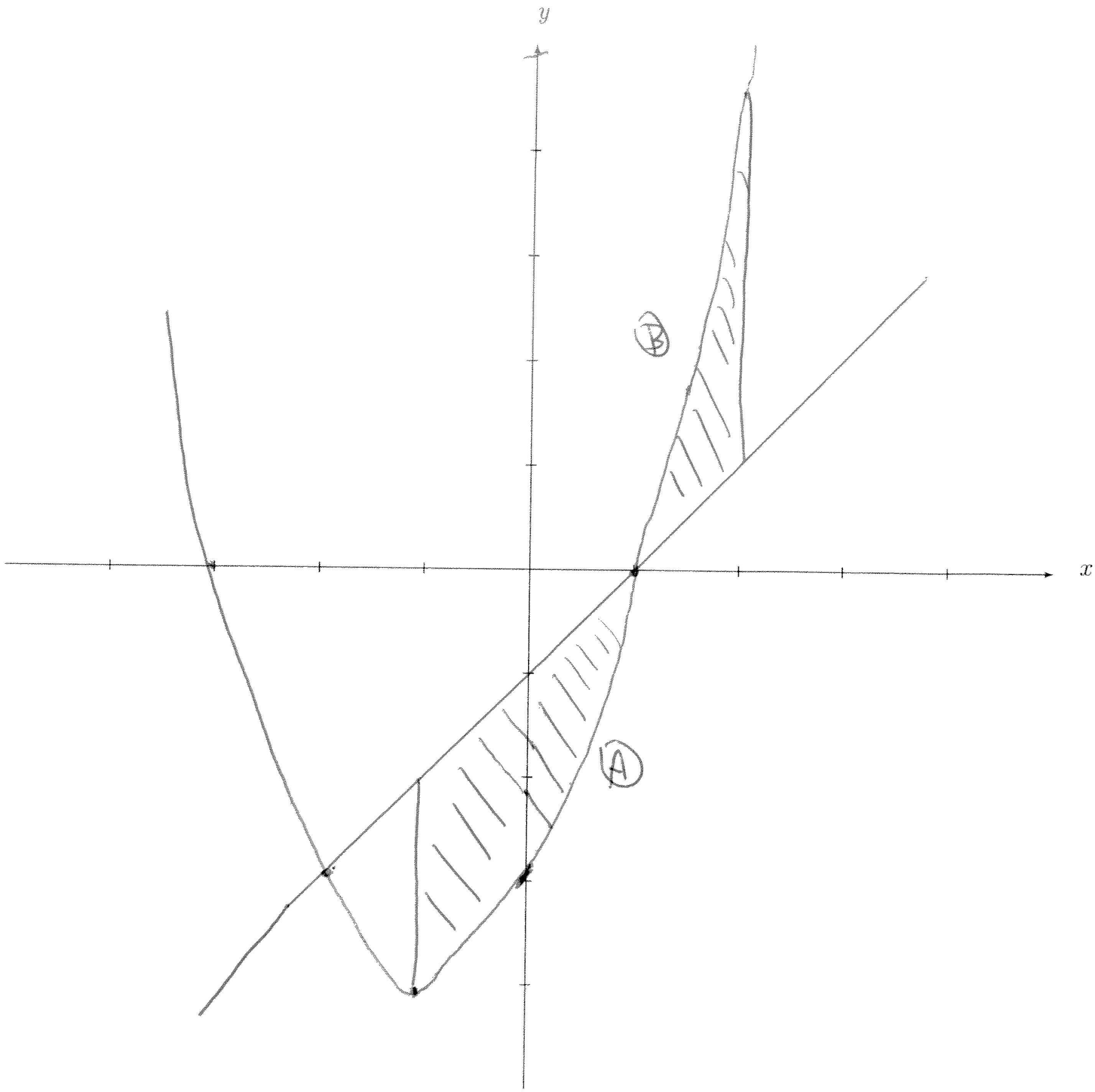
c) Area of A $= \int_{-1}^{1} [(x-1) - (x^2+2x-3)]\,dx$

$= \int_{-1}^{1} (-x^2 - x + 2)\,dx = -\frac{x^3}{3} - \frac{x^2}{2} + 2x \Big|_{-1}^{1}$

$= \left(-\frac{1}{3} - \frac{1}{2} + 2\right) - \left(\frac{1}{3} - \frac{1}{2} - 2\right) = \frac{7}{6} - \left(-\frac{13}{6}\right) = 10/3$

Area of B $= \int_{1}^{2} [(x^2+2x-3) - (x-1)]\,dx = \int_{1}^{2} (x^2 + x - 2)\,dx$

$= \frac{x^3}{3} + \frac{x^2}{2} - 2x \Big|_{1}^{2} = \left(\frac{8}{3} + 2 - 4\right) - \left(\frac{1}{3} + \frac{1}{2} - 2\right) = \frac{2}{3} - \left(-\frac{7}{6}\right) = 11/6$


y
x
B
A

Long Answer Question 5 (12 points)
Consider the function of two variables

$$f(x,y) = xy - x^3 + y^3 + 2$$

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(a) **(3 points)** Calculate the first-order partial derivatives.

(b) **(3 points)** Find all critical points.

(c) **(6 points)** Identify what type of critical points they are (local max, local min or saddle point).

$$\frac{\partial f}{\partial x} = y - 3x^2 = 0 \Rightarrow y = 3x^2$$

$$\frac{\partial f}{\partial y} = x + 3y^2 = 0$$

plug in here

$$x + 3(3x^2)^2 = x + 27x^4 = 0$$

$$\Rightarrow x(1 + 27x^3) = 0$$

$$\Rightarrow x = 0 \text{ or } x^3 = \frac{-1}{27},\ x = -1/3$$

If $x = 0$, $y = 0$
If $x = -\frac{1}{3}$, $y = 1/3$

$f_{xx} = -6x \quad f_{xy} = 1$
$f_{yy} = 6y \quad$ So $\mathcal{D} = -36xy - 1$

At $(0,0)$, $\mathcal{D} = -1$. Saddle point
At $(-1/3, 1/3)$, $\mathcal{D} = 3$ & $f_{xx} = 2$. Local min