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Lecture 3

MAT 1348 B

A compound proposition is called

- a 'tautology' if it is always True,
- a 'contradiction' if it is always False,
- a 'contingency' if it can have both True and False values.

Examples :—

tautology : $(p \wedge q) \rightarrow p$

contradiction :— $\neg((p \wedge q) \rightarrow p)$

contingency :— $(p \vee q)$

A truth table can be used to check whether a compound proposition is a tautology or a contradiction or a contingency.

Equivalence & Consistency

Equivalence :— Two propositions P & Q are said to be logically equivalent (denoted $P \equiv Q$) if $P \leftrightarrow Q$ is a tautology.

Example :— IT IS TIME FOR THE CLASS AND WE SHALL GO TO THE CLASS

Let p : "It is time for the class"

q : "We shall go to the class"

$p \wedge q \equiv \neg(\neg(p \wedge q)) \equiv \neg(\neg p \vee \neg q)$

How to Verify that a set of Propositions are equivalent :—

Equivalence can be verified using
i) truth tables
ii) basic equivalence rules
iii) truth trees

(see Page 2) Consistency can be verified using
i) truth tables
ii) truth trees

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Consistency of a set of Propositions

Given propositional variables $x, y, z, ..$ and compound propositions $P, Q, R, ..$

A set $\{P, Q, R, ...\}$ of compound propositions in the variables $x, y, z, ..$ is said to be consistent if there exists a truth assignment for $x, y, z, ...$ that makes $P, Q, R, ...$ all true at the same time.

Examples :— $P: \neg(x \wedge y)$
$Q: (x \leftrightarrow y)$
$R: \neg x \wedge \neg y$

x	y	$x \wedge y$	$P: \neg(x \wedge y)$	$Q: x \leftrightarrow y$	$R: \neg x \wedge \neg y$
F	F	F	T	T	T
F	T	F	T	F	F
T	F	F	T	F	F
T	T	T	F	T	F

So P & Q & R are consistent as ~~for~~ when x & y both are False, P & Q & R all ~~both~~ ~~have~~ are true.

How to Determine ~~Consist~~ Whether $\{P, Q, R, ...\}$ Is Consistent

- Construct a truth table for $P, Q, R, ...$
  - → columns correspond to $x, y, z, ...$ and $P, Q, R, ...$ & rows correspond to all possible truth assignments for the variables $x, y, z, ...$
- The set $\{P, Q, R, ...\}$ is consistent iff there is a row (truth assignment) in which ~~for~~ all of $P, Q, R, ...$ have True.

Satisfiability (not a contradiction)

A compound proposition P is satisfiable if there is an assignment of truth values to its variables that makes it true.

example :— $P : x \oplus y$

$Q : \neg x \wedge x \wedge y$

x	y	$\neg x$	$P : x \oplus y$	$Q : \neg x \wedge x \wedge y$
F	F	T	F	F
F	T	T	T	F
T	F	F	T	F
T	T	F	F	F

Thus P is satisfiable as there exists assignments to x, y $(x := F, y := T)$ that makes P true.

and Q is not satisfiable as for all possible assignments of x and y, Q is False.

To Determine If P is Satisfiable

- Truth Table, Truth Trees ?, Reasoning.

More examples : $(p \wedge \neg q) \vee (x \wedge \neg y \wedge z) \vee (x \wedge \neg z) \vee (p \wedge \neg x \wedge \neg z) \vee (p \wedge \neg$

Is satisfiable

Determination by reasoning : any clause is satisfiable by appropriate assignment and thus the whole proposition is satisfiable.

Facts :—

- Every tautology is also satisfiable
- satisfiability does not imply tautology
- Tautology implies ~~always~~ valid and satisfiability.

A proposition P is satisfiable if there is at least one true row in the truth table, valid if all rows are true in the truth table.

DISJUNCTIVE NORMAL FORM (DNF Formulas) OR of ANDs

- DNFs
- OR of ANDs
- Sum of Products (SOP)
- minterm

- **Atomic propositions** :- proposition containing no logical connectives.

  e.g. x, y, z

  not atomic $\neg x$, $y \wedge z$, $(x \vee y) \to \neg z$

- **Literals** :- atomic proposition or its negation

  e.g. x, $\neg x$, z

  (so if p is atomic, p and $\neg p$ are literals)

- **Conjuctive clauses** :- proposition containing only literals and possibly $\wedge$ connectives (with no atomic proposition appearing more than once)

  e.g. x, $\neg x$, $\neg x \wedge y$, $\neg x \wedge y \wedge \neg z$

  (not conjuctive examples $x \vee y$, $x \to y$, $x \to (\neg y \wedge z)$, $x \wedge \neg x$)

- **Disjunctive Normal Form** :- a proposition is said to be in disjunctive normal form (DNF) if it is a disjunction of conjuctive clause(s).

example :— p, $\neg p$, $p \wedge q$, $(x \wedge y) \vee z$, $x \vee \neg y \vee (x \wedge y \wedge \neg z)$

not DNFs :- $x \to y$, $p \vee (q \to r)$, $(x \vee y) \wedge z$

How to Find DNF Using Truth Table

- Given any compound proposition write its truth table.
- Identify the rows that have T for P.
- The disjunction of these conjective clauses is a DNF for P.

example 1 :— $(p_1 \wedge \neg p_2) \vee (p_3 \wedge p_4)$ is already in DNF.

example 2 :— $P \equiv (p_1 \to p_2) \leftrightarrow (p_1 \wedge p_2)$

p_1	p_2	$p_1 \to p_2$	$p_1 \wedge p_2$	P
F	F	T	F	F
F	T	T	F	F
T	F	F	F	T ←
T	T	T	T	T ←

p_1	p_2	P	Conjunctive Clauses
T	F	T	$p_1 \wedge \neg p_2$
T	T	T	$p_1 \wedge p_2$

The DNF formula for P is $(p_1 \wedge p_2) \vee (p_1 \wedge \neg p_2)$

Facts :— (I) DNFs are not ~~unique~~. unique. ~~[illegible]~~.

(In fact the above proposition $P \equiv p_1$)

And They are all equivalent.

— However truth table method always leads to an unique DNFs.

(II) Other methods such as truth ~~table~~ trees and logical equivalence can be used to ~~have~~ obtain DNF.

Conjuctive Normal Form

- AND of ORs
- Product of sums.
- Maxterm

Given any proposition in DNF (or any other form)

$P \equiv (\neg x \wedge y) \vee (x \wedge \neg y)$

P is in DNF

We construct the truth table

x	y	P
F	F	F ←
F	T	T
T	F	T
T	T	F ←

Now we identify those rows which has False in it.

x	y	P	Disjunctive Clauses (of negated literals)
F	F	F	$(x \vee y)$
T	T	F	$(\neg x \vee \neg y)$

Thus CNF formula is $(x \vee y) \wedge (\neg x \vee \neg y)$

$(x \vee y) \wedge (\neg x \vee \neg y)$

$\equiv (x \wedge \neg x) \vee (y \wedge \neg x) \vee (x \wedge \neg y) \vee (y \wedge \neg y)$

$\equiv F \vee (y \wedge \neg x) \vee (x \wedge \neg y) \vee F$

$\equiv (\neg x \wedge y) \vee (x \wedge \neg y)$

Further Readings (Not in Syllabus) BONUS :)

(1) • Karnaugh Map (K-map) is a method of simplifying and finding equivalent short formulas.

(2) • Mahoney Map (M Map) for dealing with larger number of variables.

(3) 3-CNF SAT (satisfiability Problem)

for example :— $(\neg x \wedge y \wedge z) \vee (x \wedge p \wedge \neg q) \vee (y \wedge \neg q \wedge m)$

Is this satisfiable ?

(4). 3 CNF SAT is NP-Complete.

(5) • The Biggest Problem In Computer Science

$$P \stackrel{?}{=} NP$$