**Question 1.** [4 points] Consider the differential equation

$$\frac{dx}{dt} = x - a^2x^3.$$

(a) Find all equilibria.

(b) Determine the stability of each equilibrium point.

(c) Draw the phase-line diagram for the differential equation.

**Solution.** a) Equilibria occur when $f(x) = x - a^2x^3 = 0$. Factoring, we have $x(1-a^2x^2) = 0$ so $x = 0, \pm\frac{1}{a}$.

- **Version 1**: $a = 2$, so the answer is $x = 0, \pm\frac{1}{2}$.
- **Version 2**: $a = 3$, so the answer is $x = 0, \pm\frac{1}{3}$.
- **Version 3**: $a = 6$, so the answer is $x = 0, \pm\frac{1}{6}$.

b) Differentiating, we have

$$f' = 1 - 3a^2x^2$$
$$f'(0) = 1 > 0$$
$$f'(\pm\frac{1}{a}) = 1 - 3a^2\frac{1}{a^2} < 0$$

Thus, $x = 0$ is unstable, while $x = \pm\frac{1}{a}$ are both stable.

c) See Figure 1.

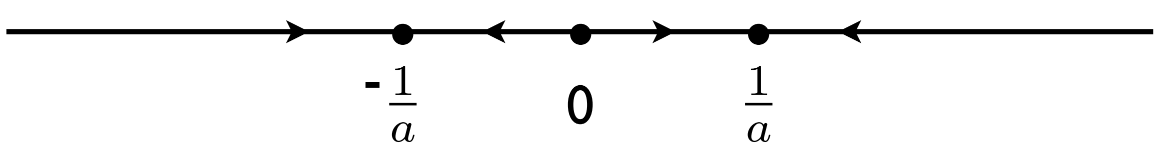


Figure 1: 1(c): $x = 0$ is unstable; the other two points are stable.

**Question 2.** [6 points] Consider the equations

- **Version 1**: $2x^3 - 8x^2 + 11x - 6 = 0$.
- **Version 2**: $2x^3 - 10x^2 + 15x - 9 = 0$.
- **Version 3**: $2x^3 - 12x^2 + 19x - 12 = 0$.

(a) Show that $x_1 = 2$ ($x_1 = 3$) [$x_1 = 4$] is a solution of the equation.

(b) Use long division to find $x_2$ and $x_3$, the other two solutions.

(c) Calculate $\bar{x}_2/x_3$, where $\bar{w}$ is the complex conjugate of $w$.

(d) Find the four roots of $x^4 = c^2$ in the form $a + ib$, where $a$ and $b$ are real.

(e) Express each root in part (d) in the form $re^{i\theta}$ with $r > 0$.

**Solution.** a)

- **Version 1**: When $x = 2$, we have $2(2^3) - 8(2^2) + 11(2) - 6 = 0$.
- **Version 2**: When $x = 3$, we have $2(3^3) - 10(3^2) + 15(3) - 9 = 0$.
- **Version 3**: When $x = 4$, we have $2(4^3) - 12(4^2) + 19(4) - 12 = 0$.

b) Long division factors the equation as

$$(x - a)(2x^2 - 4x + 3) = 0.$$

Thus, using the quadratic formula, the other two solutions satisfy

$$x = \frac{1 \pm \sqrt{16 - 4(2)(3)}}{4} = \frac{4 \pm \sqrt{8}i}{4} = 1 \pm \frac{i}{\sqrt{2}}$$

(You might notice some similarity between this and an assignment question!)

c) It actually doesn't matter which solution is $x_2$ and which is $x_3$, as the answer will be the same. If $x_2 = 1 - \frac{i}{\sqrt{2}}$, then $\bar{x}_2 = 1 + \frac{i}{\sqrt{2}}$ and thus

$$\frac{\bar{x}_2}{x_3} = \frac{1 + \frac{i}{\sqrt{2}}}{1 + \frac{i}{\sqrt{2}}} = 1$$

d) If $x^4 = c^4$, then we have $x^2 = c^2$ or $x^2 = -c^2$. From $x^2 = c^2$, we have $x = \pm c$. From $x^2 = -c^2$, we have $x = \pm ci$.

- **Version 1**: $c = 5$, so $x = \pm 5, \pm 5i$.
- **Version 2**: $c = 2$, so $x = \pm 2, \pm 2i$.
- **Version 3**: $c = 3$, so $x = \pm 3, \pm 3i$.

e) We have

$$c = ce^{i0} \qquad -c = ce^{i\pi} \qquad ci = ce^{i\pi/2} \qquad -ci = ce^{3i\pi/2}$$

- **Version 1**: $c = 5$, so

$$5 = 5e^{i0} \qquad -5 = 5e^{i\pi} \qquad 5i = 5e^{i\pi/2} \qquad -5i = 5e^{3i\pi/2}$$

- **Version 2**: $c = 2$, so

$$2 = 2e^{i0} \qquad -2 = 2e^{i\pi} \qquad 2i = 2e^{i\pi/2} \qquad -2i = 2e^{3i\pi/2}$$

- **Version 3**: $c = 3$, so

$$3 = 3e^{i0} \qquad -3 = 3e^{i\pi} \qquad 3i = 3e^{i\pi/2} \qquad -3i = 3e^{3i\pi/2}$$

**Question 3.** [5 points] Consider

- **Version 1**: $f(x, y) = \dfrac{1}{\sqrt{y + x}}$.
- **Version 2**: $f(x, y) = \dfrac{1}{\sqrt{-y - x}}$.
- **Version 3**: $f(x, y) = \dfrac{1}{\sqrt{x - y - 4}}$.

(a) Find the domain of $f$.

(b) Find the range of $f$.

(c) Sketch the domain of $f$ in the $x$-$y$ plane.

(d) On a separate graph, draw three distinct level curves.

**Solution.** a) For the function to make sense, what's under the square root can't be negative. However, it can't be zero either, because of the division. Thus, we need the thing under the square root to be positive.

- **Version 1**: The domain is $y + x > 0$, or $\{(x, y) \in \mathbb{R}^2 : y > -x\}$.
- **Version 2**: The domain is $-y - x > 0$, or $\{(x, y) \in \mathbb{R}^2 : y < -x\}$.
- **Version 3**: The domain is $x - y - 4 > 0$, or $\{(x, y) \in \mathbb{R}^2 : y < x - 4\}$.

b) The range is $f > 0$, since the square root can only produce nonnegative values, but it can't produce zero if it's in the denominator.

c) See Figure 2.

d) To find level curves, we equate the function to some value $c$. We'll take $c = 1, 2, 3$ although any three values of $c$ will be fine, except for zero.

- **Version 1**:

$$\begin{aligned} \frac{1}{\sqrt{y + x}} &= c \\ y + x &= \frac{1}{c^2} \\ y &= -x + \frac{1}{c^2} \end{aligned}$$

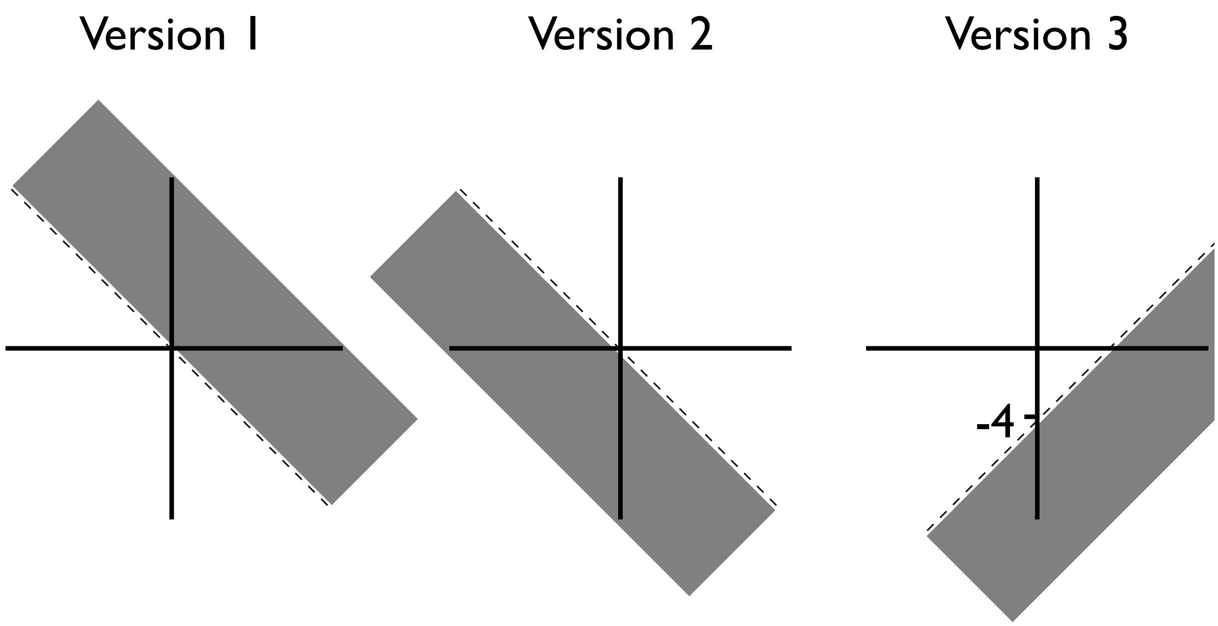


Figure 2: 3c): The shaded areas are the domains.

- **Version 2**:

$$\frac{1}{\sqrt{-y-x}} = c$$
$$-y-x = \frac{1}{c^2}$$
$$y = -x - \frac{1}{c^2}$$

- **Version 3**:

$$\frac{1}{\sqrt{x-y+4}} = c$$
$$x-y-4 = \frac{1}{c^2}$$
$$y = -x - 4 - \frac{1}{c^2}$$

See Figure 3. Note that they will be contained within the domain.

**Question 4.** [6 points] Let $A$ be the matrix

$$A = \begin{pmatrix} 0 & a \\ -1 & 2 \end{pmatrix}.$$

(a) Find both eigenvalues of $A$.
(b) For the eigenvalue with positive imaginary part, find the corresponding eigenvalues.

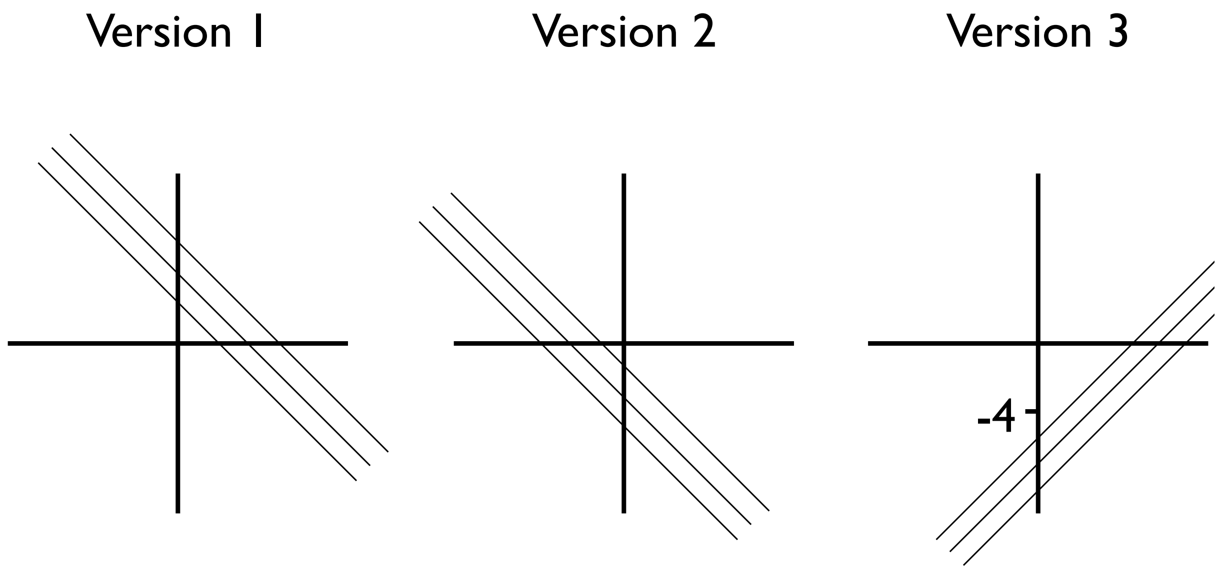


Figure 3: 3d): Three distinct level curves.

(c) State any conditions on your free variable in part (b).
(d) Show directly that $A\vec{x} = \lambda\vec{x}$ for $\lambda$ and $\vec{x}$ in part (b).

**Solution.** a)

$$\begin{aligned}
\det(A - \lambda I) &= \det \begin{pmatrix} -\lambda & a \\ -1 & 2-\lambda \end{pmatrix} \\
&= -\lambda(2-\lambda) + a \\
&= \lambda^2 - 2\lambda + a = 0 \\
\lambda &= \frac{2 \pm \sqrt{4-4a}}{2} \\
&= 1 \pm \sqrt{1-a}
\end{aligned}$$

- **Version 1**: $a = 10$, so the eigenvalues are $\lambda = 1 \pm 3i$.
- **Version 2**: $a = 5$, so the eigenvalues are $\lambda = 1 \pm 2i$.
- **Version 3**: $a = 17$, so the eigenvalues are $\lambda = 1 \pm 4i$.

b) For the eigenvalue $\lambda = 1 + bi$, we have

$$\begin{aligned}
(A - \lambda I)\vec{x} &\sim \left(\begin{array}{cc|c} -1-bi & a & 0 \\ -1 & 1-bi & 0 \end{array}\right) \\
&\sim \left(\begin{array}{cc|c} 0 & 0 & 0 \\ -1 & 1-bi & 0 \end{array}\right) \quad R_1 \to R_1 - (1+bi)R_2
\end{aligned}$$

Let $x_2 = t$. Then

$$\begin{aligned} -x_1 + (1-bi)t &= 0 \\ x_1 &= (1-bi)t \end{aligned}$$

Thus, the eigenvectors corresponding to $\lambda = 1 + bi$ are $\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} 1-bi \\ 1 \end{pmatrix} t$.

- **Version 1**: $b = 3$, so the eigenvectors are $\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} 1-3i \\ 1 \end{pmatrix} t$.
- **Version 2**: $b = 2$, so the eigenvectors are $\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} 1-2i \\ 1 \end{pmatrix} t$.
- **Version 3**: $b = 4$, so the eigenvectors are $\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} 1-4i \\ 1 \end{pmatrix} t$.

c) The free variable satisfies $t \in \mathbb{C}$, $t \neq 0$.

d)

$$\begin{aligned} A\vec{x} &= \begin{pmatrix} 0 & a \\ -1 & 2 \end{pmatrix} \begin{pmatrix} 1-bi \\ 1 \end{pmatrix} t = \begin{pmatrix} a \\ 1+bi \end{pmatrix} t \\ \lambda\vec{x} &= (1+bi) \begin{pmatrix} 1-bi \\ 1 \end{pmatrix} t = \begin{pmatrix} (1+bi)(1-bi) \\ 1+bi \end{pmatrix} t = \begin{pmatrix} a \\ 1+bi \end{pmatrix} t \end{aligned}$$

**Question 5.** [3 points] Find the complete solution to the system of equations

- **Version 1**:

$$\begin{aligned} -x + 4y + 12z &= 17 \\ 3x - y - 14z &= 4 \\ 6x + 2y - 20z &= 28. \end{aligned}$$

- **Version 2**:

$$\begin{aligned} -x + 4y - 15z &= 5 \\ 3x - y + 34z &= 18 \\ 6x + 2y + 64z &= 48. \end{aligned}$$

- **Version 3**:

$$\begin{aligned} -x + 4y + 14z &= 11 \\ 3x - y - 9z &= 11 \\ 6x + 2y - 6z &= 38. \end{aligned}$$

**Solution.** We solve these systems by writing the augmented matrix and row reducing.

- **Version 1**:

$$\left[\begin{array}{ccc|c} -1 & 4 & 12 & 17 \\ 3 & -1 & -14 & 4 \\ 6 & 2 & -20 & 28 \end{array}\right] \sim \left[\begin{array}{ccc|c} -1 & 4 & 12 & 17 \\ 0 & 11 & 22 & 55 \\ 0 & 26 & 52 & 130 \end{array}\right] \begin{array}{l} \\ R_2 \to R_2 + 3R_1 \\ R_3 \to R_3 + 6R_1 \end{array}$$

$$\sim \left[\begin{array}{ccc|c} -1 & 4 & 12 & 17 \\ 0 & 1 & 2 & 5 \\ 0 & 1 & 2 & 5 \end{array}\right] \begin{array}{l} \\ R_2 \to R_2 \div 11 \\ R_3 \to R_3 \div 26 \end{array}$$

$$\sim \left[\begin{array}{ccc|c} -1 & 4 & 12 & 17 \\ 0 & 1 & 2 & 5 \\ 0 & 0 & 0 & 0 \end{array}\right] \begin{array}{l} \\ \\ R_3 \to R_3 - R_2 \end{array}$$

Let $z = t$. Then

$$\begin{aligned} y + 2t &= 5 \\ y &= 5 - 2t \\ -x + 4(5 - 2t) + 12t &= 17 \\ x &= 3 + 4t \end{aligned}$$

Thus, the solution is $(x, y, z) = (3 + 4t, 5 - 2t, t)$.

- **Version 2**:

$$\left[\begin{array}{ccc|c} -1 & 4 & -15 & 5 \\ 3 & -1 & 34 & 18 \\ 6 & 2 & 64 & 48 \end{array}\right] \sim \left[\begin{array}{ccc|c} -1 & 4 & -15 & 5 \\ 0 & 11 & -11 & 33 \\ 0 & 26 & -26 & 78 \end{array}\right] \begin{array}{l} \\ R_2 \to R_2 + 3R_1 \\ R_3 \to R_3 + 6R_1 \end{array}$$

$$\sim \left[\begin{array}{ccc|c} -1 & 4 & -15 & 5 \\ 0 & 1 & -1 & 3 \\ 0 & 1 & -1 & 3 \end{array}\right] \begin{array}{l} \\ R_2 \to R_2 \div 11 \\ R_3 \to R_3 \div 26 \end{array}$$

$$\sim \left[\begin{array}{ccc|c} -1 & 4 & -15 & 5 \\ 0 & 1 & -1 & 3 \\ 0 & 0 & 0 & 0 \end{array}\right] \begin{array}{l} \\ \\ R_3 \to R_3 - R_2 \end{array}$$

Let $z = t$. Then

$$\begin{aligned} y - t &= 3 \\ y &= 3 + t \\ -x + 4(3 + t) - 15t &= 5 \\ x &= 7 - 11t \end{aligned}$$

Thus, the solution is $(x, y, z) = (7 - 11t, 3 + t, t)$.

- **Version 3**:

$$\left[\begin{array}{ccc|c} -1 & 4 & 14 & 11 \\ 3 & -1 & -9 & 11 \\ 6 & 2 & -6 & 38 \end{array}\right] \sim \left[\begin{array}{ccc|c} -1 & 4 & 14 & 11 \\ 0 & 11 & 33 & 44 \\ 0 & 26 & 78 & 104 \end{array}\right] \begin{array}{l} \\ R_2 \to R_2 + 3R_1 \\ R_3 \to R_3 + 6R_1 \end{array}$$

$$\sim \left[\begin{array}{ccc|c} -1 & 4 & 14 & 11 \\ 0 & 1 & 3 & 4 \\ 0 & 1 & 3 & 4 \end{array}\right] \begin{array}{l} R_2 \to R_2 \div 11 \\ R_3 \to R_3 \div 26 \end{array}$$

$$\sim \left[\begin{array}{ccc|c} -1 & 4 & 14 & 11 \\ 0 & 1 & 3 & 4 \\ 0 & 0 & 0 & 0 \end{array}\right] \begin{array}{l} \\ \\ R_3 \to R_3 - R_2 \end{array}$$

Let $z = t$. Then

$$\begin{aligned} y + 3t &= 4 \\ y &= 4 - 3t \\ -x + 4(4 - 3t) + 14t &= 11 \\ x &= 5 + 2t \end{aligned}$$

Thus, the solution is $(x, y, z) = (5 + 2t, 4 - 3t, t)$.

**Question 6.** [6 points] Consider the following matrix:

$$A = \begin{bmatrix} 1 & 0 & -c \\ 2 & 1 & -2c \\ 3 & 0 & -3c + 1 \end{bmatrix}.$$

(a) Calculate $\det(A)$.

(b) What does this imply about the invertibility of $A$?

(c) Find $A^{-1}$.

(d) Solve the equation $A\vec{x} = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$.

(e) Find the eigenvalues of $A$.

(f) Find all eigenvectors corresponding to $\lambda = 1$.

**Solution.** a)

$$\det(A) = (1)(1)(-3c + 1) + 0 + 0 - (3)(1)(-c) - 0 - 0 = 1$$

b) Since $\det(A) \neq 0$, $A$ is invertible.

c) We have

$$\left[\begin{array}{ccc|ccc} 1 & 0 & -c & 1 & 0 & 0 \\ 2 & 1 & -2c & 0 & 1 & 0 \\ 3 & 0 & -3c+1 & 0 & 0 & 1 \end{array}\right] \sim \left[\begin{array}{ccc|ccc} 1 & 0 & -c & 1 & 0 & 0 \\ 0 & 1 & 0 & -2 & 1 & 0 \\ 0 & 0 & 1 & -3 & 0 & 1 \end{array}\right] \begin{array}{l} \\ R_2 \to R_2 - 2R_1 \\ R_3 \to R_3 - 3R_1 \end{array}$$

$$\sim \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & 1-3c & 0 & c \\ 0 & 1 & 0 & -2 & 1 & 0 \\ 0 & 0 & 1 & -3 & 0 & 1 \end{array}\right] \begin{array}{l} R_1 \to R_1 + cR_3 \\ \\ \\ \end{array}$$

Thus,

$$A^{-1} = \begin{bmatrix} 1-3c & 0 & c \\ -2 & 1 & 0 \\ -3 & 0 & 1 \end{bmatrix}$$

- **Version 1**: $c = 4$, so

$$A^{-1} = \begin{bmatrix} -11 & 0 & 4 \\ -2 & 1 & 0 \\ -3 & 0 & 1 \end{bmatrix}$$

- **Version 2**: $c = 5$, so

$$A^{-1} = \begin{bmatrix} -14 & 0 & 5 \\ -2 & 1 & 0 \\ -3 & 0 & 1 \end{bmatrix}$$

- **Version 3**: $c = 7$, so

$$A^{-1} = \begin{bmatrix} -20 & 0 & 7 \\ -2 & 1 & 0 \\ -3 & 0 & 1 \end{bmatrix}$$

d) If $A$ is invertible, the equation $A\vec{x} = \vec{b}$ has solution $\vec{x} = A^{-1}\vec{b}$. Thus

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 1-3c & 0 & c \\ -2 & 1 & 0 \\ -3 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$$

$$= \begin{bmatrix} 1-2c \\ -1 \\ -2 \end{bmatrix}$$

- **Version 1**: $c = 4$, so

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} -7 \\ -1 \\ -2 \end{bmatrix}$$

- **Version 2**: $c = 5$, so

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} -9 \\ -1 \\ -2 \end{bmatrix}$$

- **Version 3**: $c = 7$, so

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} -13 \\ -1 \\ -2 \end{bmatrix}$$

e) [2 points]

$$\begin{aligned} \det(A - \lambda I) &= \det \begin{bmatrix} 1-\lambda & 0 & -c \\ 2 & 1-\lambda & -2c \\ 3 & 0 & -3c+1-\lambda \end{bmatrix} \\ &= (1-\lambda)(1-\lambda)(-3c+1-\lambda) + 0 + 0 - (3)(1-\lambda)(-c) - 0 - 0 \\ &= (1-\lambda)\big[(1-\lambda)(-3c+1-\lambda) + 3c\big] \\ &= (1-\lambda)[\lambda^2 - (2-3c)\lambda + 1] = 0 \\ \lambda &= 1, \frac{2 - 3c \pm \sqrt{(2-3c)^2 - 4}}{2} \\ &= 1, \frac{2 - 3c \pm \sqrt{9c^2 - 12c}}{2} \end{aligned}$$

- **Version 1**: $c = 4$, so

$$\lambda = 1, \frac{-10 \pm \sqrt{96}}{2} = 1, -5 \pm 2\sqrt{6}$$

- **Version 2**: $c = 5$, so

$$\lambda = 1, \frac{-13 \pm \sqrt{165}}{2}$$

- **Version 3**: $c = 7$, so

$$\lambda = 1, \frac{-19 \pm \sqrt{357}}{2}$$

f) [2 points] When $\lambda = 1$, we have $(A - I)\vec{x} = \vec{0}$. Thus

$$\left[\begin{array}{ccc|c} 0 & 0 & -c & 0 \\ 2 & 0 & -2c & 0 \\ 3 & 0 & -3c & 0 \end{array}\right]$$

From the first equation, we can see that $x_3 = 0$. Then, from the second and third equations, $x_1 = 0$. However, $x_2$ is still undetermined, so we can let $x_2 = t$. Thus, the eigenvectors are

$$\begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} t \qquad t \in \mathbb{R},\ t \neq 0.$$