Université d'Ottawa · University of Ottawa

Faculté des sciences
Mathématiques et de statistique

Faculty of Science
Mathematics and Statistics

MAT 1341A – The Midterm Test II (v.2) **Instructor: K. Zaynullin**

Last name: ______________________________

First name: ______________________________

Student number:

Please, read the following instructions carefully:

- You have 80 minutes to complete this test. **Do not detach** the pages of this examination. Read each question carefully. Where it is possible to check your work, do so.
- You can use the backs of the pages and the last page for computations.
- This is a closed book exam, and no notes of any kind are allowed. The use of programmable calculators, cell phones, laptops, pagers or any text storage or communication device is not permitted.

THIS SPACE IS RESERVED FOR THE MARKER:

Question	1	2	3	4	5	6	7	Total
Mark								
Out of	1	1	1	1	1	4	6	15

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1. Let $W = \{(x, y, z) \in \mathbf{R}^3 \mid xz = 0\}$. Then (1)

cross (X) the correct answer

- A W is closed under addition and W is closed under multiplication by scalars.
- B W is closed under addition but W is not closed under multiplication by scalars
- C W is not closed under addition but W is closed under multiplication by scalars
- D $(0, 0, 0) \in W$ but W is closed under addition
- E $(0, 0, 0) \in W$ but W is not closed under multiplication by scalars
- F None of the above is true

Solution: Let $u = (1, 0, 0)$ and $v = (0, 0, 1)$. Then both $u, v \in U$ but $u + v = (1, 0, 1) \notin U$, so it is not closed under addition. Hence, A, B, D have to be excluded.

For any $k \in \mathbf{R}$ and $u \in \mathbf{R}^3$ we have $ku = k(x, y, z) = (kx, ky, kz)$. So $(kx)(kz) = k^2xz = 0$ if $xz = 0$. In other words, if $u \in W$, then $ku \in W$. So it is closed under scalar multiplication. So E has to be excluded and the correct answer is C.

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2. Which of the following are subspaces of $\mathbb{R}^3$? (1)

$$U = \{(x, y, x + 3y) \in \mathbb{R}^3 \mid x, y \in \mathbb{R}\}$$

$$V = \{(x, y, z) \in \mathbb{R}^3 \mid x - y + 5z = 0\}$$

$$W = \{(x, y, z) \in \mathbb{R}^3 \mid xyz = 0\}$$

$$X = \{(xy, x, y) \in \mathbb{R}^3 \mid x, y \in \mathbb{R}\}$$

cross (X) the correct answer

- A Only U and W
- B Only U and V
- C Only U and X
- D Only V and W
- E Only V and X
- F Only W and X

Solution: U is a linear span of $(1, 0, 1)$ and $(0, 1, 3)$, so it is a linear subspace

V is a plane through 0, so it is a linear subspace.

W is not a plane, so it is NOT a linear subspace.

X is not a plane, so it is NOT a linear subspace.

Hence, the correct answer is B.

3. Which of the following are subspaces of the vector space $M_{2\times 2}(\mathbf{R})$? (1)

cross (X) the correct answer

- A $\left\{ \begin{pmatrix} a & b \\ c & 0 \end{pmatrix} \mid a, b, c \text{ are integers} \right\}$
- B $\left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \mid cd = 0,\ a, b, c, d \in \mathbf{R} \right\}$
- C $\left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \mid b + c = 0,\ a, b, c, d \in \mathbf{R} \right\}$
- D $\left\{ \begin{pmatrix} a & a \\ 3 & b \end{pmatrix} \mid a, b \in \mathbf{R} \right\}$
- E $\left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \mid ad = 3,\ a, b, c, d \in \mathbf{R} \right\}$
- F None of the above

Solution: D and E are not, since the zero-matrix is not of this form. A is not, since $\frac{1}{2}\begin{pmatrix} a & b \\ c & 0 \end{pmatrix}$ is not of the same form. B is not, since $\begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} + \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ 1 & 1 \end{pmatrix}$ is not of the same form. Only C gives the correct answer.

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4. Let U be a subspace of $\mathbf{R}^{11}$. Suppose U is spanned by 10 vectors and at the same time it has 8 linearly independent vectors. Then for **any** such U it is true that (1)

cross (X) the correct answer

- A $\dim U < 8$
- B $\dim U > 8$
- C $\dim U < 10$
- D $\dim U > 10$
- E $\dim U \geq 10$
- F $\dim U \geq 8$

Solution: The dimension has to be in between 8 and 10. So the correct answer is F.

5. Let u, v, w be non-zero vectors in a vector space V. Suppose that u, v, w are linearly dependent (coplanar) but at the same time v and w are linearly independent (non-colinear). Then for **any** such u, v and w the following is true (1)

cross (X) the correct answer

- [A] u and w are colinear
- [B] u and v are colinear
- [C] $v \in Span\{u, w\}$
- [D] $w \in Span\{u, v\}$
- [E] $u \in Span\{v, w\}$
- [F] None of the above

Solution: By the result proven in class the correct answer is [E].

6. Consider the vector space $\mathbf{P}_2 = \{a + bx + cx^2 \mid a, b, c \in \mathbf{R}\}$ of polynomial functions of degree at most 2. Consider the subset U of $\mathbf{P}_2$ defined by

$$U = \{f \in \mathbf{P}_2 \mid f(1) = 0\}.$$

(the subset U consists of all polynomials of degree at most 2 which have root 1)

a) Show that U is a subspace using the **Subspace Test** (1)

Solution: First, we verify $0(1) = 0$, so $0 \in U$.

Given any two polynomials $f, g \in U$ so that $f(1) = 0$ and $g(1) = 0$, we obtain

$$(f + g)(1) = f(1) + g(1) = 0 + 0 = 0.$$

So $f + g \in U$.

Finally, given any $f \in U$ (so that $f(1) = 0$) and any scalar $c \in \mathbf{R}$, we obtain

$$(cf)(1) = cf(1) = c0 = 0.$$

Hence, $cf \in U$.

b) Show that $U = Span\{1-x, x-x^2\}$ (1)

Solution: Given any polynomial $f \in U$ (so that $f(1) = 0$) we want to show that

$$f = k(1-x) + l(x-x^2) \qquad \text{for some } k, l \in \mathbf{R}.$$

Since $f(x) = a + bx + cx^2$, $f(1) = a + b + c = 0$. So that $c = -a - b$. Substituting we obtain

$$f(x) = a + bx - (a+b)x^2 = a(1-x) + (a+b)(x-x^2).$$

Hence, we can take $k = a$ and $l = a + b$.

c) Find a basis for U and explain why this is a basis

ANSWER (the basis is): (1)

Justification: (1)

Solution: Consider $1-x$ and $x-x^2$. From (b) we have $U = Span\{1-x, x-x^2\}$. We want to show that $\{1-x, x-x^2\} \subset \mathbf{P}_2$ is linearly independent.

Indeed, set $k(1-x)+l(x-x^2)=0$. Then $k-kx+lx-lx^2 = k+(l-k)x-lx^2 = 0$ which implies $k=l=0$.

So $\{1-x, x-x^2\}$ is a basis of U.

7. State whether each of the following statements is always true **(T)**, or is possibly false **(F)**, in the box after the statement.

- If you say the statement may be false, you must either give an explicit example where it fails or explain clearly why it fails.
- If you say the statement is always true, you must give a clear explanation.

a) If V is a vector space and $\{v_1, v_2, v_3, v_4\} \subset V$ is linearly dependent, then $\{v_2, v_3, v_4\} \subset V$ is also linearly dependent.

Answer (T/F): [ ] (1/2)

Justification: (1)

Solution: False.

Take $V = \mathbf{R}^4$, $v_2 = (0, 1, 0, 0)$, $v_3 = (0, 0, 1, 0)$, $v_4 = (0, 0, 0, 1)$ and $v_1 = (0, 1, 1, 1)$. Since $v_1 = v_2 + v_3 + v_4$, $\{v_1, v_2, v_3, v_4\}$ is linearly dependent. But $\{v_2, v_3, v_4\}$ is linearly independent.

b) Let v_1, v_2, v_3, v_4, v_5 be non-zero vectors in a vector space V and let $U = Span\{v_1, v_2, v_3, v_4, v_5\}$. Then $\dim U = 5$.

Answer (T/F): (1/2)

Justification: (1)

Solution: False.

Take all $v_1 = v_2 = v_3 = v_4 = v_5$. Then $\dim U = 1$.

c) If U and W are subspaces of $\mathbf{R}^3$, then their intersection

$$U \cap W = \{v \in \mathbf{R}^3 \mid v \in U \text{ and } v \in W\}$$

is closed under addition of vectors

Answer (T/F): ☐ (1/2)

Justification: (1)

Solution: True.

Let $v, u \in U \cap W$ Then $u + v \in U$ and $u + v \in W$ (as both U and W are subspaces). So $u + v \in U \cap W$.

d) Let $U = Span\{\sin^2(2x), \cos^2(2x), 4\}$ be a subspace of the vector space of all functions $F(\mathbb{R}, \mathbb{R})$. Then $\dim U = 3$.

Answer (T/F): (1/2)

Justification: (1)

Solution: False,

as $4 = 4\sin^2(2x) + 4\cos^2(2x)$, so the set $\{\sin^2(2x), \cos^2(2x), 4\}$ is linearly dependent. Hence, $\dim U < 3$.

The last page (use it for computations)