

SOLUTIONS TO MATH 1002 MID-YEAR EXAMINATION 2014

1. Indicate whether the each limit of sequences exists or does not exist. If it exists write what it equals. No proofs are required. For example,

$$\lim_{n \rightarrow \infty} 1/n \text{ exists and is equal to } 0$$

/2 (a)  $\lim_{n \rightarrow \infty} 1/n^3$  exists and equals 0

/2 (b)  $\lim_{n \rightarrow \infty} \sin\left(\frac{n\pi}{4}\right)$  does not exist

/2 (c)  $\lim_{n \rightarrow \infty} n^{1/n}$  exists and equals 1

/2 (d)  $\lim_{n \rightarrow \infty} \frac{n^3 + 2}{n^2 - n + 1}$  exists and is equal to  $\infty$

2. Indicate whether each sequence below is monotone or not. If it is monotone indicate whether it is increasing or decreasing. No justification is required.

/2 (a)  $(n + 2(-1)^n)_{n=1}^{\infty}$  not monotone

/2 (b)  $(\tan(2n\pi + \pi/4))_{n=1}^{\infty}$  monotone and increasing (or decreasing since constant)

3. Provide the limit of each function below. No computation or justification is required.

/1 (a)  $\lim_{x \rightarrow 7} \pi = \pi$  (constant function)

/1 (b)  $\lim_{x \rightarrow \infty} 2x^2 - x - 1 = \infty$

/1 (c)  $\lim_{x \rightarrow 0^-} \frac{|x|}{x} = \lim_{x \rightarrow 0^-} \frac{-x}{x} = -1$

/1 (d)  $\lim_{x \rightarrow 4} \frac{4x - x^2}{2 - \sqrt{x}} = \lim_{x \rightarrow 4} \frac{x(2 - \sqrt{x})(2 + \sqrt{x})}{2 - \sqrt{x}} = 16$

/1 (e)  $\lim_{x \rightarrow 0} \frac{e^{2x} - \cos(x)}{x} = \lim_{x \rightarrow 0} \frac{2e^{2x} + \sin(x)}{1} = 2$  (l'Hôpital's Rule)

4. Define  $f(x) = (2x^2 + 1)^2(x^3 - x + 1)^3$  for all  $x \in \mathbb{R}$ .

/2 (a) Compute the derivative  $f'$ . Show your work.

**Solution:** Using product and chain rule, we compute  $f'(x) = (2(2x^2 + 1))(4x)(x^3 - x + 1)^3 + (2x^2 + 1)^2(3(x^3 - x + 1)^2(3x^2 - 1))$

/4 (b) Explain why  $f$  has at least one root and provide a bounded interval in which this root may be found.

**Solution:**  $f$  is a polynomial so it is continuous everywhere. Observe that  $f(1) = 9$  and  $f(-1) = -9$ . This means that the restriction of  $f$  to  $[-1, 1]$  is a continuous function on a bounded interval with  $f(-1) < 0 < f(1)$ . By the Intermediate Value Theorem there exists a root  $c \in (-1, 1)$  (i.e.  $f(c) = 0$ ).

**Grading:** 1 for continuity. 1 for  $f(-1) < 0 < f(1)$ . 1 for IVT. 1 for interval.

/6 5. Using only theorems from the text, prove the following limit of a sequence

$$\lim_{n \rightarrow \infty} \frac{n^2 + 6}{n + 1} = \infty.$$

No theorem numbers are required, but try to make clear which theorems you are using and why you may use them.

**Solution:**

$$\frac{n^2 + 6}{n + 1} = \frac{n^2(1 + 6/n^2)}{n(1 + 1/n)} = n \left( \frac{(1 + 6/n^2)}{1 + 1/n} \right)$$

By Thm 9.3  $\lim 1 + 1/n = \lim 1 + \lim 1/n = 1 + 0$  (as both limits converge.)

By Thm 9.4  $\lim 1/n^2 = (\lim 1/n)(\lim 1/n) = (0)(0)$  (again both limits converge). By Thms 9.3 and 9.2  $\lim 1 + 6/n^2 = \lim 1 + 6 \lim 1/n^2 = 1 + 6(0) = 1$ .

By Thm 9.6  $\lim_{n \rightarrow \infty} \frac{(1+6/n^2)}{1+1/n} = 1/1 = 1$  (limit in denominator converges to  $1 \neq 0$ ).

By Thm 9.10  $\lim_{n \rightarrow \infty} n = \lim_{n \rightarrow \infty} \frac{1}{1/n} = \infty$ , since  $n > 0$  and  $\lim_{n \rightarrow \infty} 1/n = 0$ .  
Finally, by Thm 9.9

$$\lim_{n \rightarrow \infty} \frac{n^2 + 6}{n + 1} = \lim_{n \rightarrow \infty} n \left( \frac{(1 + 6/n^2)}{1 + 1/n} \right) = \infty$$

since the  $\lim_{n \rightarrow \infty} \frac{(1+6/n^2)}{1+1/n} = 1 > 0$ .

**Grading:** 1 for putting the quotient into the right form. 2 for giving limit of numerator or denominator. 1 for giving limit of  $\frac{(1+6/n^2)}{1+1/n}$ . 2 for final limit.

- /5 6. Let  $s_n = \sin(n\pi/2)$  for all  $n \in \mathbb{N}$ . Define a subsequence  $(t_k)$  of  $(s_n)$  which is a constant sequence. Justify that your definition is a constant subsequence. (Hint: Replace the question mark in  $t_k = \sin(?\pi/2)$  with the right thing.)

**Solution:** Let  $t_k = \sin(2k\pi/2) = \sin(k\pi) = 0$  for all  $k \in \mathbb{N}$ . In this way,  $t_k = s_{2k}$  for all  $k \in \mathbb{N}$ . Moreover,  $2k < 2k + 2 = 2(k + 1)$  for all  $k \in \mathbb{N}$ . Clearly  $t_k$  is the constant sequence  $(0)_{k=1}^{\infty}$ .

**Grading:** 2 for definition of  $t_k$ . 1 for identifying constant. 2 for proof of strictly increasing indices.

- /3 7. Suppose  $(a_k)_{k=1}^{\infty}$  is a sequence. Provide the definition for the infinite series  $\sum_{k=1}^{\infty} a_k$  to converge. (Not the Cauchy Criterion.)

**Solution:**  $\sum_{k=1}^{\infty} a_k$  converges if

$$\lim_{n \rightarrow \infty} \sum_{k=1}^n a_k$$

converges.

8. Determine which of the following series converge. Justify your answers by using any tests or series we have covered in this course.

/3 (a)  $\sum_{n=1}^{\infty} \frac{n^4}{2^n}$

**Solution:** We use the Ratio Test. First

$$\left| \frac{(n+1)^4/2^{n+1}}{n^4/2^n} \right| = \left( \frac{n+1}{n} \right)^4 \left( \frac{2^n}{2^{n+1}} \right) = \frac{(1+1/n)^4}{2}$$

Moreover  $\lim_{n \rightarrow \infty} \frac{(1+1/n)^4}{2} = (1/2)(1 + \lim_{n \rightarrow \infty} 1/n)^4 = 1/2 < 1$  so that the series converges (absolutely).

**Solution:** 1 for correct test. 1 for computation. 1 for correct conclusion.

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(b)  $\sum_{n=1}^{\infty} \frac{2}{n}$

**Solution:** We know that the harmonic series satisfies  $\sum_{n=1}^{\infty} \frac{2}{n} = \infty$ . Since  $2/n > 1/n$  for all  $n \in \mathbb{N}$  the Comparison Test tells us that the above series also diverges (to infinity).

**Grading:** 1 for divergence of harmonic series. 1 for correct test. 1 for correct conclusion.

/3

(c)  $\sum_{n=1}^{\infty} \left( \frac{n+1}{3n+2} \right)^n$

**Solution:** We use the Root Test

$$\left( \left( \frac{n+1}{3n+2} \right)^n \right)^{1/n} = \frac{n+1}{3n+2}$$

and

$$\lim_{n \rightarrow \infty} \frac{n+1}{3n+2} = \lim_{n \rightarrow \infty} \frac{1+1/n}{3+2/n} = 1/3 < 1$$

Therefore the series converges (absolutely).

**Grading:** See part (a).

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9. Suppose  $f : \mathbb{R} \rightarrow \mathbb{R}$  is a function and  $a \in \mathbb{R}$ . Provide the definition of what it means for  $f$  to be differentiable at  $a$ .

**Solution:** This means that  $\lim_{x \rightarrow a} \frac{f(x) - f(a)}{x - a}$  exists and is a real number (equal to  $f'(a)$ ).

/2

10. Let  $f(x) = x$  for all  $x \in \mathbb{R}$ . Using only the definition above, prove that  $f$  is differentiable at  $a \in \mathbb{R}$ .

**Solution:** Suppose  $a \in \mathbb{R}$ . Then

$$\frac{x - a}{x - a} = 1, \quad x \neq a$$

and so  $\lim_{x \rightarrow a} \frac{x - a}{x - a} = 1$ .

11. Let  $n \in \mathbb{N}$  and  $g(x) = x^n$ . Prove by induction on  $n$  that  $g$  is differentiable at  $a \in \mathbb{R}$ . You may use rules for differentiability without proof.

/4

**Solution:** Base case: Suppose  $n = 1$ . Then  $g(x) = x$  for all  $x \in \mathbb{R}$  and  $g$  is differentiable at  $a \in \mathbb{R}$  by the previous proof.

Induction assumption: Assume  $g$  is differentiable at  $a \in \mathbb{R}$ .

Induction step: Suppose now that  $h(x) = x^{n+1}$  for all  $x \in \mathbb{R}$  and  $k(x) = x$  for all  $x \in \mathbb{R}$ . Then  $h = gk$ . By the induction assumption (and the base case)  $g$  and  $k$  are differentiable at  $a \in \mathbb{R}$ . Therefore  $h$  is differentiable at  $a \in \mathbb{R}$  (Theorem 28.3 (iii)).

**Grading:** 1 for base case. 1 for induction assumption. 2 for induction step and use of induction assumption.

12. Suppose  $f : (0, 1) \rightarrow \mathbb{R}$  is differentiable and  $f'(x) > 0$  for all  $x \in (0, 1)$ . Explain how we get a contradiction to the Mean Value Theorem if we assume that  $f$  is decreasing on some interval  $(a, b) \subseteq (0, 1)$ .

/5

**Solution:** Assume that  $f$  is decreasing on  $(a, b) \subseteq (0, 1)$ . This means that for any  $a \leq a_1 < b_1 \leq b$  we have  $f(a_1) \geq f(b_1)$ . Choose  $a_1$  and  $b_1$  as above with  $a < a_1$  and  $b_1 < b$ . Then  $f$  is differentiable at every  $c \in [a_1, b_1]$  and is therefore also continuous at every  $c \in (a_1, b_1)$ . We may therefore apply MVT to the restriction of  $f$  to the closed interval  $[a_1, b_1]$ . This gives us

$$f'(c) = \frac{f(b_1) - f(a_1)}{b_1 - a_1}$$

for some  $c \in (a_1, b_1)$ . Now  $c \in (0, 1)$  so that  $f'(c) > 0$ . However,  $f(b_1) - f(a_1) \leq 0$  and  $b_1 - a_1 > 0$  imply that  $\frac{f(b_1) - f(a_1)}{b_1 - a_1} \leq 0$ . This is a contradiction.

**Grading:** 2 for adapting to the hypotheses in the statement of MVT (continuity and differentiability). 1 for applying MVT. 2 for contradiction.

/6

13. Suppose  $a, M \in \mathbb{R}$ . Prove that  $|a| < M$  if and only if  $-M < a < M$ .

**Solution:**

Suppose  $|a| < M$ . Then  $M \geq 0$  since  $|a| \geq 0$  and  $-M \leq 0$ . If  $a \geq 0$  then  $-M \leq 0 < a = |a| < M$  so  $-M < a < M$ . If  $a < 0$  then  $-a = |a| < M$  implies  $-M < -|a| = a$  and also  $a < 0 \leq M$ . In this case then we also have  $-M < a < M$

Now we prove the converse. Suppose  $-M < a < M$ . If  $a \geq 0$  then  $|a| = a < M$ . If  $a < 0$  then  $|a| = -a < -(-M) = M$ .

**Grading:** 1 for style. 3 for first implication above and 2 for the second.

/4

14. Suppose  $S \subseteq \mathbb{R}$  is a non-empty subset which is bounded above and bounded below. Prove that there is an  $M \in \mathbb{R}$  such that  $|a| < M$  for all  $a \in S$ . (Hint: Use the result of the previous question.)

**Solution:**  $S$  bounded above means there is  $M_1 \in \mathbb{R}$  such that  $M_1 > a$  for all  $a \in S$ .  $S$  bounded below means there is  $M_2 \in \mathbb{R}$  such that  $M_2 < a$  for all  $a \in S$ . Without loss of generality, we may assume that  $M_1 > 0$  and we may write  $M_2 = -M_3$  where  $M_3 > 0$ . Let  $M = \max\{M_1, M_2\}$ . Then  $M \geq M_1$  so that  $M > a$  for all  $a \in S$ . Similarly  $M \geq M_3$  implies in turn that  $-M \leq -M_3 < a$  for all  $a \in S$ . In conclusion  $-M \leq a \leq M$  for all  $a \in S$  and we may apply the previous exercise to conclude that  $|a| < M$  for all  $a \in S$ .

**Grading:** 1 for definition of bounded above/below. 1 for taking max of something. 2 for showing  $-M \leq a \leq M$ .

/5

15. Provide a proof from the definition of limit that  $\lim_{x \rightarrow 2} 3x - 1 = 5$ . (Hint: There must be deltas and epsilons in your proof.)

**Solution:** Suppose  $\epsilon > 0$ . Let  $\delta = \epsilon/3$ . Then  $\delta > 0$  and if  $0 < |x - 2| < \delta$  we have

$$\begin{aligned} |x - 2| < \epsilon/3 &\Rightarrow 3|x - 2| < \epsilon \\ &\Rightarrow |3x - 6| < \epsilon \\ &\Rightarrow |3x - 1 - 6 + 1| < \epsilon \\ &\Rightarrow |(3x - 1) - 5| < \epsilon \end{aligned}$$

**Grading:** 2 for style. 1 for correct  $\delta$ . 2 for correct proof.

/6

16. Suppose  $a, L \in \mathbb{R}$ ,  $f : \mathbb{R} \rightarrow \mathbb{R}$  and  $g : \mathbb{R} - \{a\} \rightarrow \mathbb{R}$  satisfy  $\lim_{x \rightarrow a} g(x) = L$  and  $\lim_{x \rightarrow L} f(x) = f(L)$ . Prove that

$$\lim_{x \rightarrow a} f \circ g(x) = f(L).$$

(Hint: Careful!  $g$  is *not* continuous at  $a$ .)

**Solution:** The following are true: (\*) For all  $\epsilon_1 > 0$  there exists  $\delta_1 > 0$  such that

$$0 < |x - a| < \delta_1 \Rightarrow |g(x) - L| < \epsilon_1$$

(\*\*) For all  $\epsilon_2 > 0$  there exists  $\delta_2 > 0$  such that

$$0 < |x - L| < \delta_2 \Rightarrow |f(x) - f(L)| < \epsilon_2$$

Suppose  $\epsilon > 0$ . Taking  $\epsilon_2 = \epsilon$  in (\*\*), we obtain  $\delta_2 > 0$  such that

$$0 < |x - L| < \delta_2 \Rightarrow |f(x) - f(L)| < \epsilon. \quad (1)$$

Taking  $\epsilon_1 = \delta_2$  in (\*) we obtain  $\delta_1 > 0$  such that

$$0 < |x - a| < \delta_1 \Rightarrow |g(x) - L| < \delta_2. \quad (2)$$

Let  $\delta = \delta_1$ . Then

$$0 < |x - a| < \delta = \delta_1 \Rightarrow |g(x) - L| < \delta_2 \Rightarrow |f(g(x)) - f(L)| < \epsilon$$

by (1) and (2).

**Grading:** 2 for choosing  $\epsilon_2 = \epsilon$ . 2 for choosing  $\epsilon_1 = \delta_2$ . 2 for connecting the two implications.

/3

17. Suppose  $f : \mathbb{R} \rightarrow \mathbb{R}$  is a function. Define what it means for  $f$  to be one-to-one (or injective).

**Solution:**  $f(a) = f(b)$  for  $a, b \in \mathbb{R}$  implies that  $a = b$ .

18. Let  $f(x) = \cos(x)$  for all  $x \in [0, \pi]$ .

/4

(a) Prove that  $f$  is a strictly decreasing function. (Hint: Take the derivative.)

**Solution:**  $f'(x) = -\sin(x)$  for all  $x \in (0, \pi)$  and  $-\sin(x) < 0$  for all  $x \in (0, \pi)$ . Therefore  $f$  is strictly decreasing on  $(0, \pi)$ . Moreover  $f(0) = 1 > f(x) > -1 = f(\pi)$  for all  $x \in (0, \pi)$  so that  $f$  is strictly decreasing on  $[0, \pi]$

**Grading:** 3 for the derivative being less than zero. 1 for the endpoints.

- (b) Explain why  $f$  has an inverse function which is differentiable on some open interval.

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**Solution:**  $f$  is one-to-one. Otherwise there are  $0 \leq a < b \leq \pi$  such that  $f(a) = f(b)$  and this contradicts  $f$  being strictly increasing. We know that  $f$  is invertible if and only if  $f$  is one-to-one. Now,  $f^{-1} : [-1, 1] \rightarrow \mathbb{R}$  is differentiable on  $(-1, 1) = f((0, \pi))$  since  $f$  is differentiable on  $(0, \pi)$  and  $f'(x) \neq 0$  for all  $x \in (0, \pi)$ .

**Grading:** 2 for existence of inverse. 1 for differentiability

19. Indicate whether each of the following statements is true or false. Provide some justification for your choice in each case.

- (a) Let  $f : \mathbb{R} \rightarrow \mathbb{R}$ . If  $\lim_{n \rightarrow \infty} s_n = s$  and  $\lim_{n \rightarrow \infty} f(s_n) = f(s)$  then  $f$  is continuous at  $s$ .

/3

**Solution:** False. Suppose  $f(x) = 1$  for  $x \geq 0$ ,  $f(x) = -1$  for  $x < 0$ , and  $s_n = 1/n$  for  $n \in \mathbb{N}$ . Then  $\lim f(1/n) = 1 = f(0)$ , but  $f$  is discontinuous at 0.

- (b) Let  $f(x) = x^{25} + \sin(1/x)$  for all  $x \in [1, 2]$ . The function  $f$  is uniformly continuous.

/3

**Solution:**  $f$  is a continuous function on a bounded and closed interval. It is therefore uniformly continuous.

- (c) The sequence  $\left( \frac{\cos(n)}{|\sin(n)|+1} \right)_{n=1}^{\infty}$  has a convergent subsequence.

/3

**Solution:** True.  $\left| \frac{\cos(n)}{|\sin(n)|+1} \right| \leq \frac{1}{|\sin(n)|+1} \leq 1$  so that the sequence is bounded. By the Bolzano-Weierstrass Theorem, it must have a convergent subsequence.