Solution to Final Examination

MAT1341E, Fall 2018

Part I. Multiple-choice Questions (1 × 10 = 10 marks)

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1. Let **F** be the vector space of all real-valued functions defined for all real numbers with ordinary addition and scalar multiplication operations. Which two of the following subsets of **F** are subspaces of **F**?

$S = \{f \in \mathbf{F} \mid f(-2)f(2) = 0\}$;
$T = \{f \in \mathbf{F} \mid f(0) = 0\}$;
$U = \{f \in \mathbf{F} \mid f(1) = f(0)\}$;
$V = \{f \in \mathbf{F} \mid f(1) = -1\}$.

(A) S and T; (B) S and U; (C) S and V; (D) T and U; (E) T and V; (F) U and V.

Solution. (D) S is not a subspace. Let $f_1(-2) = 1$, $f_1(2) = 0$, and let $f_2(-2) = 0$, $f_2(2) = 1$. Then f_1 and f_2 are in S, but $(f_1 + f_2)(-2) = (f_1 + f_2)(2) = 1$, and $f_1 + f_2$ is not S. S is not closed under addition.

T is a subspace. Since the zero function $f_0(x) = 0$ with $f_0(0) = 0$ is in T. If f is in T, then $(cf)(0) = c(f(0) = 0$. T is closed under scalar multiplication. If f_1 and f_2 are in T, then $f_1(0) = 0$, and $f_2(0) = 0$, and $(f_1 + f_2)(0) = f_1(0) + f_2(0) = 0$. Hence $f_1 + f_2$ is in T. T is closed under addition.

U is a subspace. Since the zero function $f_0(x) = 0$ with $f_0(1) = f_0(0) = 0$ is in U. If f is in U, then $(cf)(0) = c(f(0) = c(f(1) = (cf)(1)$. T is closed under scalar multiplication. If f_1 and f_2 are in U, then $f_1(1) = f_1(0)$, and $f_2(1) = f_2(0)$, and $(f_1 + f_2)(1) = f_1(1) + f_2(1) = = f_1(0) + f_2(0) = (f_1 + f_2)(0) = 0$. Hence $f_1 + f_2$ is in T. T is closed under addition.

V is not a subspace since the zero function $f_0(x) = 0$ is not in V.

2. Suppose $S = \{\mathbf{u}, \mathbf{v}, \mathbf{w}\}$ is a set of vectors in a vector space V. Which two of the following statements is equivalent to the statement that "S is linearly independent"?

(1) None of the vectors **u**, **v**, or **w** is a linear combination of the other two vectors in S.
(2) None of the vectors **u**, **v**, or **w** is a multiple of any one of the other two vectors in S.
(3) If a, b and c are scalars and $a\mathbf{u} + b\mathbf{v} + c\mathbf{w} = 0$, then we must have $a = b = c = 0$.
(4) If $a = b = c = 0$, then $a\mathbf{u} + b\mathbf{v} + c\mathbf{w} = 0$.

(A) (1) and (2); (B) (1) and (3); (C) (1) and (4);
(D) (2) and (3); (E) (2) and (4); (F) (3) and (4).

Solution. (B) (1) is true. If there exist one vector in S, say $\mathbf{u}$, is a liner combination of $\mathbf{v}$ and $\mathbf{w}$, then $\mathbf{u} = a\mathbf{v} + b\mathbf{w}$. We have a dependence relation $\mathbf{u} - a\mathbf{v} - c\mathbf{w} = \mathbf{0}$, where not all coefficients are zero. Then S is linearly dependent.

(2) is false. In set $S = \{(1, 0), (1, 1), (2, 1)\}$, none of these vectors is a multiple of any of the other two, but S is linearly dependent as $(1, 0) + (1, 1) = (2, 1)$.

(3) is true. This is the definition of linear independence.

(4) is false. This equality is always true no matter this set is linearly dependent or linearly independent.

3. Let A be an $n \times n$ square matrix with $n \geq 2$. Which one(s) of the following statements are true?

(1) If rank $(A) = 1$, then there is exactly one parameter in the general solution of the homogeneous system $A\mathbf{x} = \mathbf{0}$.
(2) If rank $(A) = 1$, then there are exactly $n - 1$ parameters in the general solution of the homogeneous system $A\mathbf{x} = \mathbf{0}$.
(3) If A is invertible, then the homogeneous system $A\mathbf{x} = \mathbf{0}$ has a unique solution.
(4) If the homogeneous system $A\mathbf{x} = \mathbf{0}$ has infinitely many solutions, then rank $(A) = n$.

(A) (1) only; (B) (2) only; (C) (1) and (3); (D) (2) and (3);
(E) (1) and (4); (F) (2) and (4).

Solution. (D) The number of parameters in the general solution of the homogeneous system $A\mathbf{x} = \mathbf{0}$ equals n – rank (A). Hence, (1) is false, and (2) is true. If A is invertible, then rank $(A) = n$. Hence, the number of parameters is zero, and the system has a unique solution. Then (3) is true, and (4) is false..

4. The polar form of complex number $z = \dfrac{1-\sqrt{3}i}{i-1}$ is

(A) $\sqrt{2}\left(\cos\left(-\frac{7}{12}\pi\right)+i\sin\left(-\frac{7}{12}\pi\right)\right)$; (B) $\sqrt{2}\left(\cos\left(\frac{11}{12}\pi\right)+i\sin\left(\frac{11}{12}\pi\right)\right)$;

(C) $\sqrt{2}\left(\cos\left(\frac{5}{12}\pi\right)+i\sin\left(\frac{5}{12}\pi\right)\right)$; (D) $\sqrt{2}\left(\cos\left(\frac{1}{12}\pi\right)+i\sin\left(\frac{1}{12}\pi\right)\right)$;

(E) $\sqrt{2}\left(\cos\left(-\frac{1}{12}\pi\right)+i\sin\left(-\frac{1}{12}\pi\right)\right)$; (F) $\sqrt{2}\left(\cos\left(-\frac{5}{12}\pi\right)+i\sin\left(-\frac{5}{12}\pi\right)\right)$.

Solution. (B) Since all choices have the same modulus $|z| = \sqrt{2}$, and all the arguments are between $-\pi$ and π, what we have to do is to determine Arg z.

Since $z_1 = 1-\sqrt{3}i = 2\left(\frac{1}{2}-\frac{\sqrt{3}}{2}i\right) = 2\left(\cos\left(-\frac{\pi}{3}\right)+i\sin\left(-\frac{\pi}{3}\right)\right)$, and $z_2 = -1+i = \sqrt{2}\left(-\frac{1}{\sqrt{2}}+\frac{1}{\sqrt{2}}i\right) = \sqrt{2}\left(\cos\left(\frac{3\pi}{4}\right)+i\sin\left(\frac{3\pi}{4}\right)\right)$, $\arg z_1 = -\frac{\pi}{3} + 2n\pi$, and $\arg z_2 = \frac{3\pi}{4} + 2n\pi$. Hence, $\arg z = \left(-\frac{\pi}{3}-\frac{3\pi}{4}\right)+2n\pi = -\frac{13}{12}\pi + 2n\pi$, and $\text{Arg } z = \frac{11\pi}{12}$.

5. Let $M_{3\times 3}$ be the vector space of real matrices of dimensions 3×3 with ordinary matrix addition and scalar multiplication operations. $U = \{A \in M_{3\times 3} \mid A^T = -A\}$ is a subspace of $M_{3\times 3}$. Then dim (U) =

(A) 0; (B) 2; (C) 3; (D) 4; (E) 6; (F) 9.

Solution. (C) Let $A = \begin{bmatrix} a & b & c \\ d & e & f \\ g & h & i \end{bmatrix}$. Then $A^T = -A$ gives $\begin{bmatrix} a & d & g \\ b & e & h \\ c & f & i \end{bmatrix} = \begin{bmatrix} -a & -b & -c \\ -d & -e & -f \\ -g & -h & -i \end{bmatrix}$. This means that $a = -a$, $e = -e$, and $i = -i$. Hence, $a = e = i = 0$. Since $d = -b$, $g = -c$, and $h = -f$, the set of matrices that have the property $A^T = -A$ is

$$\begin{bmatrix} 0 & b & c \\ -b & 0 & f \\ -c & -f & 0 \end{bmatrix} = b\begin{bmatrix} 0 & 1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} + c\begin{bmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ -1 & 0 & 0 \end{bmatrix} + f\begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & -1 & 0 \end{bmatrix}.$$ Then $U = \text{span } S$, where

$$S = \left\{\begin{bmatrix} 0 & 1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ -1 & 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & -1 & 0 \end{bmatrix}\right\}.$$ It is easy to see that S is linearly independent. Then S is a basis of U, and dim $(U) = 3$.

6. Let $A = \begin{bmatrix} 1 & 1 & 2 \\ 1 & 0 & 1 \\ 2 & 1 & 3 \end{bmatrix}$. Which one of the following statements is true?

(A) A is not invertible.
(B) The third row of A^{-1} is $(-1, -1, 1)$.
(C) The second row of A^{-1} is $(1, 2, -1)$.
(D) The first row of A^{-1} is $(2, 0, -1)$.
(E) The second column of A^{-1} is $(0, 2, -1)^T$.
(F) All of (B), (C), (D), and (E) are true.

Solution. (A) Try to find the inverse of A. Reduce the following matrix to RREF:

$$\begin{bmatrix} 1 & 1 & 2 & 1 & 0 & 0 \\ 1 & 0 & 1 & 0 & 1 & 0 \\ 2 & 1 & 3 & 0 & 0 & 1 \end{bmatrix} \sim \begin{bmatrix} 1 & 1 & 2 & 1 & 0 & 0 \\ 0 & -1 & -1 & -1 & 1 & 0 \\ 0 & -1 & -1 & -2 & 0 & 1 \end{bmatrix} \sim \begin{bmatrix} 1 & 1 & 2 & 1 & 0 & 0 \\ 0 & 1 & 1 & 1 & -1 & 0 \\ 0 & -1 & -1 & -2 & 0 & 1 \end{bmatrix}$$
$$\sim \begin{bmatrix} 1 & 1 & 2 & 1 & 0 & 0 \\ 0 & 1 & 1 & 1 & -1 & 0 \\ 0 & 0 & 0 & -1 & -1 & 1 \end{bmatrix}.$$

We see rank $(A) = 2 < 3$. Hence, A is not invertible.

Another easier way to see this is to notice that the third row is the sum of the first two rows. Hence, the set of rows of this matrix is linearly dependent. It is not invertible.

7. Let $\mathbf{u} = (0, 3, 4)$, $\mathbf{v} = (1, 0, 0)$, and $\mathbf{w} = (0, 4, -3)$. (Note that $\{\mathbf{u}, \mathbf{v}, \mathbf{w}\}$ is an orthogonal set). If $\mathbf{x} = (0, -1, -1) = a\mathbf{u} + b\mathbf{v} + c\mathbf{w}$, then $(a, b, c) =$

(A) $\left(-\frac{7}{5}, 0, -\frac{1}{5}\right)$; (B) $\left(-\frac{7}{25}, 0, -\frac{1}{25}\right)$; (C) $\left(\frac{7}{25}, 0, \frac{1}{25}\right)$;
(D) $\left(-\frac{7}{25}, 0, \frac{1}{5}\right)$; (E) $(-7, 0, -1)$; (F) $(0, -1, -1)$.

Solution. (B) We can solve this problem by solving a system of equations

$(0, -1, -1) = a(0, 3, 4) + b(1, 0, 0) + c(0, 4, -3)$.

However, since this set is orthogonal, it is easier to use the Fourier coefficients:

$$a = \frac{\mathbf{x}\cdot\mathbf{u}}{\|\mathbf{u}\|^2} = \frac{0+(-3)+(-4)}{0^2+(-3)^2+(-4)^2} = -\frac{7}{25}.$$
$$b = \frac{\mathbf{x}\cdot\mathbf{v}}{\|\mathbf{v}\|^2} = \frac{0+0+0}{1^2+0^2+0^2} = 0.$$
$$c = \frac{\mathbf{x}\cdot\mathbf{w}}{\|\mathbf{w}\|^2} = \frac{0+(-4)+3}{0^2+4^2+(-3)^2} = -\frac{1}{25}.$$

8. Consider the matrix $A = \begin{bmatrix} 3 & 1 \\ 0 & 3 \end{bmatrix}$. What are the answers to the following questions (Yes/No)?

(1) Is 3 the only eigenvalue of A?
(2) Is the dimension of the eigenspace corresponding to eigenvalue 3 equal to 1?
(3) Is A diagonalizable?

(A) Yes, Yes, Yes. (B) Yes, Yes, No. (C) No, Yes, Yes.
(D) No, Yes, No. (E) No, No, Yes. (F) Yes, No, No.

Solution. (B) Let det $(A - \lambda I) = \begin{vmatrix} 3-\lambda & 1 \\ 0 & 3-\lambda \end{vmatrix} = (3 - \lambda)^2 = 0$. Hence, $\lambda = 3$ is the only eigenvalue of A.

When $\lambda = 3$, solve the system $\begin{bmatrix} 0 & 1 \\ 0 & 1 \end{bmatrix}\mathbf{v} = \mathbf{0}$. The general solution to this system is $\mathbf{v} = \begin{bmatrix} 1 \\ 0 \end{bmatrix} t$.
The dimension of the eigenspace corresponding to eigenvalue 3 is 1. Since the algebraic dimension of this eigenvalue is less than the geometric dimension of this eigenvalue, A is not diagonalizable.

9. If $\begin{vmatrix} a & b & c \\ d & e & f \\ g & h & i \end{vmatrix} = 3$, then $\begin{vmatrix} 4g & a & d-2a \\ 4h & b & e-2b \\ 4i & c & f-2c \end{vmatrix} =$

(A) 6; (B) −6; (C) 12; (D) −12; (E) 24; (F) −24.

Solution. (C) See Practice 4, #2.

$$\begin{vmatrix} 4g & a & d-2a \\ 4h & b & e-2b \\ 4i & c & f-2c \end{vmatrix} = 4\begin{vmatrix} g & a & d-2a \\ h & b & e-2b \\ i & c & f-2c \end{vmatrix} = 4\begin{vmatrix} g & a & d \\ h & b & e \\ i & c & f \end{vmatrix} = -4\begin{vmatrix} a & g & d \\ b & h & e \\ c & i & f \end{vmatrix}$$
$$= 4\begin{vmatrix} a & d & g \\ b & e & h \\ c & f & i \end{vmatrix} = 4\begin{vmatrix} a & b & c \\ d & e & f \\ g & h & i \end{vmatrix} = 12.$$

10. Let **F** be the vector space of all real-valued function defined for all real numbers with ordinary addition and scalar multiplication operations. Which two of the following subsets of **F** is linearly independent?

$S = \{\sin x, \cos x\}$; $T = \{1, \sin x, \cos x\}$;
$U = \{1, \sin^2 x, \cos^2 x\}$; $V = \{1, 2\sin^2 x, 3\cos^2 x\}$.

(A) S and T; (B) S and U; (C) S and V; (D) T and U; (E) T and V; (F) U and V.

Solution. (A) See Midterm 2, #6 (d).

Since $\sin^2 x + \cos^2 x = 1$, $\frac{1}{2}(2\sin^2 x) + \frac{1}{3}(3\cos^2 x) = 1$, U and V are linearly dependent. Since there are only two of them are linearly independent, we must have S and T linearly independent.

Part II. Long Answer Questions (14 + 2 marks)

11. (3 marks) Consider the system of linear equations

$$\begin{cases} kx + 2y + \ \ z = 0, \\ \qquad y + \ \ z = 0, \\ \qquad\quad 3z = 0, \end{cases}$$

where k is a real constant, and x, y and z are unknowns. Find all value(s) of k, if any, such that this system

(a) has a unique solution; (b) has infinitely many solutions; (c) has no solution.

Solution. See Practice 1, #6.

This is a homogeneous system. It must be consistent. There is no value of k to make this system to have no solution. The coefficient matrix of this system is $A = \begin{bmatrix} k & 2 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 3 \end{bmatrix}$. When $k = 0$, rank(A) = 2, and the system has infinitely many solutions. When $k \neq 0$, rank(A) = 3, and this system has a unique solution.

12. (3 marks) Let $\mathbf{v}_1 = (1, 0, 0, -1)$, $\mathbf{v}_2 = (1, -1, 0, 0)$, $\mathbf{v}_3 = (1, 0, 1, 0)$. Consider the subspace $U = \text{span}\,\{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3\}$ of $\mathbb{R}^4$.

(a) Show that $\{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3\}$ is a basis of U.

Solution. See Midterm 2, #7 (c).

It is enough to show that $\{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3\}$ is linearly independent.

Let $x(1, 0, 0, -1) + \underline{y}(1, -1, 0, 0) + z(1, 0, 1, 0) = (x + y + z, -y, z, -x) = (0, 0, 0, 0)$. Then $x = y = z = 0$.

(b) Use the Gram-Schmidt algorithm to find an orthogonal basis of U.

Solution. See practice 3, #6 (a).

Let $\mathbf{u}_1 = \mathbf{v}_1 = (1, 0, 0, -1)$.

$\mathbf{v}_2 - \dfrac{\mathbf{v}_2 \cdot \mathbf{u}_1}{\|\mathbf{u}_1\|^2}\mathbf{u}_1 = (1,-1,0,0) - \dfrac{1}{2}(1,0,0,-1) = \dfrac{1}{2}(1,-2,0,1)$. Let $\mathbf{u}_2 = (1, -2, 0, 1)$.

$$\mathbf{v}_3 - \frac{\mathbf{v}_3 \cdot \mathbf{u}_1}{\|\mathbf{u}_1\|^2}\mathbf{u}_1 - \frac{\mathbf{v}_3 \cdot \mathbf{u}_2}{\|\mathbf{u}\|^2}\mathbf{u}_2 = (1,0,1,0) - \frac{1}{2}(1,0,0,-1) - \frac{1}{6}(1,-2,0,1) = \frac{1}{3}(1,1,3,1).$$

Let $\mathbf{u}_3 = (1, 1, 3, 1)$

$\{\mathbf{u}_1, \mathbf{u}_2, \mathbf{u}_3\}$ is an orthogonal basis of U.

(c) Find a vector in U that is the best approximation of the vector $\mathbf{v} = (0, 1, 1, 1)$.

Solution. See Practice #6 (b).

The best approximation of $\mathbf{v}$ in U is

$$\text{proj}_U\,\mathbf{v} = \frac{\mathbf{v}\cdot\mathbf{u}_1}{\|\mathbf{u}_1\|^2}\mathbf{u}_1 + \frac{\mathbf{v}\cdot\mathbf{u}_2}{\|\mathbf{u}\|^2}\mathbf{u}_2 + \frac{\mathbf{v}\cdot\mathbf{u}_2}{\|\mathbf{u}\|^2}\mathbf{u}_2$$
$$= -\frac{1}{2}(1,0,0,-1) - \frac{1}{6}(1,-2,0,1) + \frac{5}{12}(1,1,3,1) = \frac{1}{12}(-3,9,15,9) = \frac{1}{4}(-1,3,5,3).$$

If you cannot do part (b), you can still find the projection without using an orthogonal basis:

See Practice 3, #7.

Let $A = \begin{bmatrix} 1 & 1 & 1 \\ 0 & -1 & 0 \\ 0 & 0 & 1 \\ -1 & 0 & 0 \end{bmatrix}$, and let $\mathbf{b} = \begin{bmatrix} 0 \\ 1 \\ 1 \\ 1 \end{bmatrix}$. Then $A^T = \begin{bmatrix} 1 & 0 & 0 & -1 \\ 1 & -1 & 0 & 0 \\ 1 & 0 & 1 & 0 \end{bmatrix}$, $A^T\mathbf{b} = \begin{bmatrix} -1 \\ -1 \\ 1 \end{bmatrix}$, and $A^TA = \begin{bmatrix} 2 & 1 & 1 \\ 1 & 2 & 1 \\ 1 & 1 & 2 \end{bmatrix}$. Solve the system $A^TA\mathbf{x} = A^T\mathbf{b}$, i.e., $\begin{bmatrix} 2 & 1 & 1 \\ 1 & 2 & 1 \\ 1 & 1 & 2 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} -1 \\ -1 \\ 1 \end{bmatrix}$.

Reduce the augmented matrix to RREF:

$$\begin{bmatrix} 2 & 1 & 1 & -1 \\ 1 & 2 & 1 & -1 \\ 1 & 1 & 2 & 1 \end{bmatrix} \sim \begin{bmatrix} 1 & 1 & 2 & 1 \\ 1 & 2 & 1 & -1 \\ 2 & 1 & 1 & -1 \end{bmatrix} \sim \begin{bmatrix} 1 & 1 & 2 & 1 \\ 0 & 1 & -1 & -2 \\ 0 & -1 & -3 & -3 \end{bmatrix} \sim \begin{bmatrix} 1 & 1 & 2 & 1 \\ 0 & 1 & -1 & -2 \\ 0 & 0 & -4 & -5 \end{bmatrix} \sim \begin{bmatrix} 1 & 1 & 2 & 1 \\ 0 & 1 & -1 & -2 \\ 0 & 0 & 1 & 5/4 \end{bmatrix}$$

$$\sim \begin{bmatrix} 1 & 1 & 0 & -6/4 \\ 0 & 1 & 0 & -3/4 \\ 0 & 0 & 1 & 5/4 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & 0 & -3/4 \\ 0 & 1 & 0 & -3/4 \\ 0 & 0 & 1 & 5/4 \end{bmatrix}. \quad \mathbf{x} = \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \frac{1}{4}\begin{bmatrix} -3 \\ -3 \\ 5 \end{bmatrix}.$$

Then $\text{proj}_U\, \mathbf{v} = A\mathbf{x} = \frac{1}{4}\begin{bmatrix} 1 & 1 & 1 \\ 0 & -1 & 0 \\ 0 & 0 & 1 \\ -1 & 0 & 0 \end{bmatrix}\begin{bmatrix} -3 \\ -3 \\ 5 \end{bmatrix} = \frac{1}{4}\begin{bmatrix} -1 \\ 3 \\ 5 \\ 3 \end{bmatrix}$.

13. (4 marks) Let $A = \begin{bmatrix} 1 & 0 & -2 \\ 1 & 3 & 1 \\ 1 & 0 & 4 \end{bmatrix}$.

(a) Use the factorized characteristic polynomial of A to show that A has only two eigenvalues 2 and 3.

Solution. See Practice 4, #4.

Let $\det(A - \lambda I) = \det \begin{bmatrix} 1-\lambda & 0 & -2 \\ 1 & 3-\lambda & 1 \\ 1 & 0 & 4-\lambda \end{bmatrix} = 0$. Expend the determinant down the second column. We have

$$(3-\lambda)\begin{vmatrix} 1-\lambda & -2 \\ 1 & 4-\lambda \end{vmatrix} = (3-\lambda)((1-\lambda)(4-\lambda)+2) = (3-\lambda)(\lambda^2 - 5\lambda + 6) = (3-\lambda)(\lambda-3)(\lambda-2)$$

$= (3-\lambda)^2(2-\lambda) = 0$. Hence $\lambda = 2$ and $\lambda = 3$ are only eigenvalues of A.

(b) Find a basis of the eigenspace corresponding to the eigenvalue 2.

Solution. When $\lambda = 2$, solve the system $(A - 2I)\mathbf{v} = \mathbf{0}$. Reduce the coefficient matrix $A - 2I = \begin{bmatrix} -1 & 0 & -2 \\ 1 & 1 & 1 \\ 1 & 0 & 2 \end{bmatrix}$ to RREF:

$\begin{bmatrix} -1 & 0 & -2 \\ 1 & 1 & 1 \\ 1 & 0 & 2 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & 2 \\ 1 & 1 & 1 \\ 1 & 0 & 2 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & 2 \\ 0 & 1 & -1 \\ 0 & 0 & 0 \end{bmatrix}$. The general form of eigenvectors is $\begin{bmatrix} -2 \\ 1 \\ 1 \end{bmatrix}t$, where $t \neq 0$. A basis of the eigenspace is $\left\{\begin{bmatrix} -2 \\ 1 \\ 1 \end{bmatrix}\right\}$.

(c) Find a basis of the eigenspace corresponding to the eigenvalue 3.

Solution. When $\lambda = 3$, solve the system $(A - 3I)\mathbf{v} = \mathbf{0}$. Reduce the coefficient matrix $A - 3I = \begin{bmatrix} -2 & 0 & -2 \\ 1 & 0 & 1 \\ 1 & 0 & 1 \end{bmatrix}$ to RREF:

$\begin{bmatrix} -2 & 0 & -2 \\ 1 & 0 & 1 \\ 1 & 0 & 1 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & 1 \\ 1 & 0 & 1 \\ 1 & 0 & 1 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$. The general form of eigenvectors is $\begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} s + \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix} t$, where $t \neq 0$. A basis of the eigenspace is $\left\{ \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix} \right\}$.

(d) Find an invertible matrix P and a diagonal matrix D such that $P^{-1}AP = D$. (I don't want P^{-1}!)

Solution. An eigenvector basis of $\mathbb{R}^3$ is $B = \left\{ \begin{bmatrix} -2 \\ 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix} \right\}$. Then $P = \begin{bmatrix} -2 & 0 & -1 \\ 1 & 1 & 0 \\ 1 & 0 & 1 \end{bmatrix}$, and $D = \begin{bmatrix} 2 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 3 \end{bmatrix}$.

14. (4 marks) Determine whether each of the following statements is always true (T) justified by a proof using results presented in classes, or possibly false (F) disproved by a counter-example:

(a) If $\mathbf{u}$, $\mathbf{v}$, and $\mathbf{w}$ are three linearly independent vectors in a vector space V, then $\mathbf{u} \in \text{span}\,\{\mathbf{u} + \mathbf{v}, \mathbf{v} + \mathbf{w}\}$.

Answer. F

Justification or counter-example.

Let $\mathbf{u} = (1, 0, 0)$, $\mathbf{v} = (0, 1, 0)$, and $\mathbf{w} = (0, 0, 1)$. Then $\{\mathbf{u}, \mathbf{v}, \mathbf{w}\}$ is linearly independent. $\mathbf{u} + \mathbf{v} = (1, 1, 0)$ and $\mathbf{v} + \mathbf{w} = (0, 1, 1)$. If $\mathbf{u} = x(\mathbf{u} + \mathbf{v}) + y(\mathbf{v} + \mathbf{w})$, then $(1, 0, 0) = x(1, 1, 0) + y(0, 1, 1) = (x, x + y, y)$. We have $x = 1$, $x + y = 0$, $y = 0$. This is impossible.

(b) Let A be a 4×3 matrix with real entries. If the system $A\mathbf{x} = \mathbf{0}$ has a non-trivial solution, then the set of columns of A is linearly dependent.

Answer. T

Justification or counter-example.

Let $A = [\mathbf{c}_1\ \mathbf{c}_2\ \mathbf{c}_3]$, where $\mathbf{c}_i$ is the i-th column of A. Let $\mathbf{x} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$ be a non-trivial solution of the homogeneous system $A\mathbf{x} = \mathbf{0}$. Then $A\mathbf{x} = x_1\mathbf{c}_1 + x_2\mathbf{c}_2 + x_3\mathbf{c}_3 = \mathbf{0}$. The set $\{\mathbf{c}_1, \mathbf{c}_2, \mathbf{c}_3\}$ is linearly dependent.

(c) Let A be a 2×3 matrix. Then every vector in Null (A) is orthogonal to every row of A.

Answer. T

Justification or counter-example.

Let $A = \begin{bmatrix} \mathbf{r}_1 \\ \mathbf{r}_2 \end{bmatrix}$, where $\mathbf{r}_i$ is the i-th row of A. If $\mathbf{x} \in$ Null(A), $A\mathbf{x} = \begin{bmatrix} \mathbf{r}_1 \\ \mathbf{r}_2 \end{bmatrix}\mathbf{x} = \mathbf{0}$. Hence, $\mathbf{r}_1\mathbf{x} = \mathbf{0}$, and $\mathbf{r}_2\mathbf{x} = \mathbf{0}$. Vector $\mathbf{x}$ is orthogonal to both rows of A.

(d) If A is an invertible 3×3 matrix, then A is diagonalizable.

Answer. F

Justification or counter-example.

See Practice 4, # 5.

Let $A = \begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$. Since det ($A$) = 1, A is invertible. det $(A - \lambda I) = (1 - \lambda)^3$. $\lambda = 1$ is the only eigenvalue of A with algebraic multiplicity 3. When $\lambda = 1$, $A - I = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$. The eigenspace is $\left\{ \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} s + \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} t \right\}$. The geometric multiplicity of eigenvalue $\lambda = 1$ is 2. A is not diagonolizable.

15. (Bonus 2 marks) Define a linear transformation T: $\mathbb{R}^3 \to \mathbb{R}^3$:

$$T\left(\begin{bmatrix} x \\ y \\ z \end{bmatrix}\right) = \begin{bmatrix} x+y \\ y-z \\ z+x \end{bmatrix}.$$

(a) Find the standard matrix of T.

See Example in Notes, Chapter 24, 5.

Solution. $T\left(\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}\right) = \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}$, $T\left(\begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}\right) = \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}$, $T\left(\begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}\right) = \begin{bmatrix} 0 \\ -1 \\ 1 \end{bmatrix}$.

The standard matrix is $A = \begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & -1 \\ 1 & 0 & 1 \end{bmatrix}$.

(b) Find a basis of the image space im(T).

Solution. The image space im (T) is the column space of the standard matrix: im(T) = Col(A).

Use the column space algorithm to find a basis of Col(A). Reduce A to an REF:

$$\begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & -1 \\ 1 & 0 & 1 \end{bmatrix} \sim \begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & -1 \\ 0 & -1 & 1 \end{bmatrix} \sim \begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & -1 \\ 0 & 0 & 0 \end{bmatrix}.$$

Since each of the first two columns has a leading one, the first two columns of A form a basis of Col(A) = im (T). I.e., $B = \left\{ \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} \right\}$ is a basis of im (T).