Lec Review

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Ex 1 $y'' + 3y' - 4y = -2 + 4\delta(t-3)$; $y(0) = 1$; $y'(0) = -1$

Sol Let $Y(s) = \mathcal{L}\{y(t)\}$. Apply $\mathcal{L}$ to the ODE:

$$\mathcal{L}\{y''\} + 3\mathcal{L}\{y'\} - 4\mathcal{L}\{y\} = -2\mathcal{L}\{1\} + 4\mathcal{L}\{\delta(t-3)\} \Rightarrow$$

$$s^2Y - sy(0) - y'(0) + 3[sY - y(0)] - 4Y = -\frac{2}{s} + 4e^{-3s}$$

$$s^2Y - s + 1 + 3sY - 3 - 4Y = \frac{-2}{s} + 4e^{-3s}$$

$$(s^2 + 3s - 4)Y = s + 2 - \frac{2}{s} + 4e^{-3s}$$

$$Y = \frac{s+2}{s^2+3s-4} - \frac{2}{s(s^2+3s-4)} + \frac{4e^{-3s}}{s^2+3s-4}$$

$$y(t) = \mathcal{L}^{-1}\{Y(s)\} = \mathcal{L}^{-1}\left\{\frac{s+2}{s^2+3s-4}\right\} - \mathcal{L}^{-1}\left\{\frac{2}{s(s^2+3s-4)}\right\} + 4\mathcal{L}^{-1}\left\{\frac{e^{-3s}}{s^2+3s-4}\right\}$$

- For $\mathcal{L}^{-1}\left\{\frac{s+2}{s^2+3s-4}\right\}$:

$$\frac{s+2}{s^2+3s-4} = \frac{s+2}{(s-1)(s+4)} = \frac{A}{s-1} + \frac{B}{s+4} = \frac{(A+B)s + 4A - B}{(s-1)(s+4)}$$

$\Rightarrow A + 1 = 1$ ① ①+②: $5A = 3$

$4A - B = 2$ ② $\underline{A = 3/5}$

$$\underline{B = 2/5}$$

$$\frac{s+2}{s^2+3s-4} = \frac{3}{5}\frac{1}{s-1} + \frac{2}{5}\frac{1}{s+4} \Rightarrow \frac{3}{5}\mathcal{L}^{-1}\left\{\frac{1}{s-1}\right\} + \frac{2}{5}\mathcal{L}^{-1}\left\{\frac{1}{s+4}\right\}$$

$$\boxed{\mathcal{L}^{-1}\left\{\frac{s+2}{s^2+3s-4}\right\} = \frac{3}{5}e^{t} + \frac{2}{5}e^{-4t}}$$

- For $\mathcal{L}^{-1}\left\{\frac{2}{s(s^2+3s-4)}\right\}$

$$\frac{2}{s(s^2+3s-4)} = \frac{2}{s(s-1)(s+4)} = \frac{A}{s} + \frac{B}{s-1} + \frac{C}{s+4}$$

$$= \frac{A(s^2+3s-4)+B(s^2+4s)+C(s^2-s)}{s(s-1)(s+4)}$$

$$\Rightarrow \left.\begin{array}{ll} A+B+C=0 & (1) \\ 3A+4B-C=0 & (2) \\ -4A=2 & (3) \end{array}\right\} \Rightarrow \underline{A=-\frac{1}{2}}$$

(1)+(2):

$$4A+5B=0$$

$$4\left(-\frac{1}{2}\right)+5B=0$$

$$\underline{B=\frac{2}{5}}$$

$$-\frac{1}{2}+\frac{2}{5}+C=0$$

$$\underline{C=\frac{1}{2}-\frac{2}{5}=\frac{5}{10}-\frac{4}{10}=\frac{1}{10}}$$

$$\frac{2}{s(s^2+3s-4)} = \left(-\frac{1}{2}\right)\frac{1}{s} + \frac{2}{5}\frac{1}{s-1} + \frac{1}{10}\frac{1}{s+4}$$

$$\boxed{\mathcal{L}^{-1}\left\{\frac{2}{s(s^2+3s-4)}\right\} = -\frac{1}{2} + \frac{2}{5}e^{t} + \frac{1}{10}e^{-4t}}$$

• For $\mathcal{L}^{-1}\left\{\frac{e^{-3s}}{s^2+3s-4}\right\} = \mathcal{L}^{-1}\left\{e^{-3s}\underbrace{\frac{1}{s^2+3s-4}}_{F(s)}\right\}$

$$\mathcal{L}^{-1}\{e^{-as}F(s)\} = u(t-a)f(t-a)$$

$$f(t) = \mathcal{L}^{-1}\{F(s)\} = \mathcal{L}^{-1}\left\{\frac{1}{s^2+3s-4}\right\}$$

$$\frac{1}{s^2+3s-4} = \frac{A}{s-1} + \frac{B}{s+4} = \frac{(A+B)s+4A-B}{s^2+3s-4} \Rightarrow \begin{array}{l} A+B=0 \\ 4A-B=1 \end{array} \Rightarrow \begin{array}{l} A=\frac{1}{5} \\ B=-\frac{1}{5} \end{array}$$

So $\mathcal{L}^{-1}\left\{\frac{1}{s^2+3s-4}\right\} = \frac{1}{5}\mathcal{L}^{-1}\left\{\frac{1}{s-1}\right\} - \frac{1}{5}\mathcal{L}^{-1}\left\{\frac{1}{s^2+4}\right\} = \frac{1}{5}e^{t} - \frac{1}{5}e^{-4t}$

By second shifting theorem:

$$\boxed{\mathcal{L}^{-1}\left\{\frac{e^{-3s}}{s^2+3s-4}\right\} = u(t-3)\left[\frac{1}{5}e^{t-3} - \frac{1}{5}e^{-4(t-3)}\right]}$$

So the solution to the IVP is

$$y(t) = \frac{1}{5}e^{t} + \frac{3}{10}e^{-4t} + \frac{1}{2} + \frac{4}{5}u(t-3)\left[e^{t-3} - e^{(-4t+12)}\right]$$

~~Ex 2~~ Find $\mathcal{L}\{\underbrace{(u(t-3)\cdot(t^2+t-1))}_{f(t)} * \underbrace{e^{-3t}\cos(2t)}_{g(t)}\}$

$$\underbrace{}_{f(t)} \qquad \underbrace{}_{g(t)}$$

Sol

By the convolution theorem: $\mathcal{L}\{f(t) * g(t)\} = \mathcal{L}\{f(t)\} \cdot \mathcal{L}\{g(t)\}$

- For $\mathcal{L}\{u(t-3)\, \underbrace{(t^2+t-1)}_{=f(t-3)}\}$

$$f(t-3) = t^2+t-1 \Rightarrow f(t) = (t+3)^2 + (t+3) - 1$$
$$= t^2 + 7t + 11$$

$$\mathcal{L}\{u(t-3)(t^2+t-1)\} = e^{-3s}\,\mathcal{L}\{t^2+7t+11\} \text{ (second shifting theorem)}$$
$$= e^{-3s}\left[\frac{2}{s^3} + \frac{7}{s^2} + \frac{11}{s}\right]$$

- $\underline{\mathcal{L}\{e^{-3t}\cos(2t)\}}$

First

$$F(s) = \mathcal{L}\{\cos(2t)\} = \frac{s}{s^2+4}$$

By first shifting theorem:

$$\mathcal{L}\{e^{-3t}\cos(2t)\} = \frac{s+3}{(s+3)^2+4} = \frac{s+3}{s^2+6s+13}$$

$$\mathcal{L}\left\{e^{-3t}\cos(2t)\right\} = \frac{s+3}{(s+3)^2+4} = \frac{s+3}{s^2+6s+13}$$

So $\mathcal{L}\left\{\left(u(t-3)\cdot(t^2+t-1)\right) * e^{-3t}\cos(2t)\right\}$

$$\boxed{e^{-3s}\left(\frac{2}{s^3}+\frac{7}{s^2}+\frac{11}{s}\right)\cdot\frac{s+3}{s^2+6s+13}}$$

Ex Find $\mathcal{L}^{-1}\left\{\frac{e^{-2s}}{s^2+4s+5}\right\} = \mathcal{L}^{-1}\left\{e^{-2s}\ \frac{1}{s^2+4s+5}\right\}$

$$f(t) = \mathcal{L}^{-1}\left\{\frac{1}{s^2+4s+5}\right\} = \mathcal{L}^{-1}\left\{\frac{1}{(s^2+4s+4)+1}\right\}$$

$$= \mathcal{L}^{-1}\left\{\frac{1}{(s+2)^2+1}\right\} = \sin t\ e^{-2t}$$

So $\mathcal{L}^{-1}\left\{\frac{e^{-2s}}{s^2+4s+5}\right\} = u(t-2)\,e^{-2(t-2)}\,\sin(t-2)$