Lec 23 Extra

November 30, 2018 10:00 AM

Last time

Division by t Rule

$$F(s) = \mathcal{L}\{f(t)\}$$

If $\lim\limits_{t \to 0} \dfrac{f(t)}{t}$ exists, then

$$\boxed{\mathcal{L}\left\{\frac{f(t)}{t}\right\} = \int_s^{+\infty} F(x)\,dx}$$

Ex 1 Find $\mathcal{L}\left\{\dfrac{e^t - 1}{t}\right\}$

Sol $\lim\limits_{t \to 0} \dfrac{e^t - 1}{t} = \dfrac{0}{0}$

$$\overset{L'H}{=} \lim_{t \to 0} \frac{e^t}{1} = 1$$

$$F(s) = \mathcal{L}\{e^t - 1\} = \mathcal{L}\{e^t\} - \mathcal{L}\{1\} = \frac{1}{s-1} - \frac{1}{s}$$

$$\therefore \mathcal{L}\left\{\frac{e^t - 1}{t}\right\} = \int_s^{+\infty} \left(\frac{1}{x-1} - \frac{1}{x}\right) dx = \lim_{L \to \infty} \int_s^L \left(\frac{1}{x-1} - \frac{1}{x}\right) dx$$

$$= \lim_{L \to \infty} \Big[\ln(x-1) - \ln x\Big]_0^L = \lim_{L \to \infty} \Big[\ln|(L-1)| - \ln|(L)|\Big] - \Big[\ln|(s-1)| - \ln|(s)|\Big]$$

L'H

0

$$= \lim_{L\to\infty}\left[\ln\left(\cancel{\frac{L-1}{L}}\right)^{\overset{L'H}{\downarrow}\to 0} - \ln\left(\frac{s-1}{s}\right)\right] = -\ln\left(\frac{s-1}{s}\right)^{-1} = \ln\left(\frac{s}{s-1}\right)$$

$$\boxed{\mathcal{L}\left\{\frac{e^t-1}{t}\right\} = \ln\left(\frac{s}{s-1}\right)}$$

Ex Find $\mathcal{L}\left\{\frac{\sin t}{t}\right\}$

<u>Sol</u> $\lim_{t\to 0}\left(\frac{\sin t}{t}\right) \overset{L'H}{=} \lim_{t\to 0}\left(\frac{\cos t}{1}\right) = 1$ exists ✓

$\mathcal{L}\{\sin t\} = \frac{1}{s^2+1}$ change "s" for "x"

$$\mathcal{L}\left\{\frac{\sin t}{t}\right\} = \int_s^\infty \frac{1}{x^2+1}\,dx = \lim_{L\to\infty}\left[\tan^{-1}x\right]_s^L$$

$$= \lim_{L\to\infty}\left[\tan^{-1}(L) - \tan^{-1}(s)\right] = \frac{\pi}{2} - \tan^{-1}(s)$$

Laplace of an integral

Very often, the function we deal with are given in terms of an integral

$$g(t) = \int_0^t f(x)\,dx$$

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Even if we don't know how to integrate $\int f(x)\,dx$, we still can compute $\mathcal{L}\{g(t)\}$.

$$\underbrace{g(t) = \int_0^t f(x)\,dx}_{} \Rightarrow g'(t) = f(t)$$

(Fundamental theorem of calculus)

integral of $g(t)$ between $\emptyset$ and $\emptyset$

$$g'(t) = f(t) \Rightarrow \mathcal{L}\{g'(t)\} = \mathcal{L}\{f(t)\}$$

$$= s\,\mathcal{L}\{g(t)\} - \cancel{g(0)}^{\,0} = \mathcal{L}\{f(t)\}$$

$$s\,\underbrace{\mathcal{L}\{g(t)\}}_{} = \mathcal{L}\{f(t)\} \Rightarrow \mathcal{L}\{g(t)\} = \frac{1}{s}\mathcal{L}\{f(t)\}$$

So $\boxed{\mathcal{L}\left\{\int_0^t f(x)\,dx\right\} = \frac{1}{s}\mathcal{L}\{f(t)\}}$

Ex $\mathcal{L}\left\{\int_0^t \sin(2x)\,dx\right\} = \frac{1}{s}\mathcal{L}\{\sin(2t)\}$

$$= \frac{1}{s}\left(\frac{2}{s^2+4}\right)$$

La guerre Equation

Let $n \in \mathbb{Z}$ (a positive or negative integer)

An ODE of the form:

$$\boxed{t y'' + (1-t) y' + n y = 0}$$

is called La <u>guerre Equation</u>

We solve it using Laplace

Let $Y(s) = \mathcal{L}\{y(t)\}$

Apply $\mathcal{L}$ to both sides:

$$\mathcal{L}\{t y''\} + \mathcal{L}\{y'\} - \mathcal{L}\{t y'\} + n \mathcal{L}\{y\} = 0$$

Use multiplication by t Rule

$$= (-1)^1 \frac{d}{ds} \mathcal{L}\{y''\} + s \mathcal{L}\{y\} - y(0) - (-1)^1 \frac{d}{ds} \mathcal{L}\{y'\} + n \mathcal{L}\{y\} = 0$$

$$-\frac{d}{ds}\left[s^2 Y - s y(0) - y'(0)\right] + sY - y(0) + \frac{d}{ds}\left[sY - y(0)\right] + nY = 0$$

Constant

$$-\left[2sY + s^2 Y' - y(0)\right] + sY - y(0) + \left[Y + sY'\right] + nY = 0$$

$$-2sY - s^2Y' + y(0) + sY - y(0) + Y + sY' + nY = 0$$

$$(-s+1+n)Y = (s^2-s)Y'$$

$$(-s+1+n)Y = (s^2-s)\frac{dY}{ds} \Rightarrow \frac{(-s+1+n)}{s^2-s}ds = \frac{1}{Y}dY$$ Separable

$$\int \frac{-s+1+n}{s^2-s}ds = \int \frac{1}{Y}dY$$ — $\ln|Y|$ $(\cancel{+c})$

For $\int \frac{-s+1+n}{s^2-s}ds$ use partial fractions

$$\frac{-s+1+n}{s(s-1)} = \frac{A}{s} + \frac{B}{s-1} \Rightarrow \begin{cases} A+B = -1 \\ -A = n+1 \end{cases}$$

$$A = 1-n \Rightarrow B = -1+n+1 = n$$

So $\frac{-s+n+1}{s^2-s} = \frac{-n-1}{s} + \frac{n}{s-1}$

$$\int \frac{-s+n+1}{s^2-s}ds = -\int \frac{n+1}{s}ds + n\int \frac{1}{s-1}ds$$

$$= -(n+1)\ln(s) + n\ln(s-1)$$

$$= \ln(s-1)^n - \ln(s^{n+1})$$

$$= \ln\left(\frac{(s-1)^n}{s^{n+1}}\right)$$

back to (*) :

$$\ln\left(\frac{(s-1)^n}{s^{n+1}}\right) = \ln(Y)$$

$$Y = \frac{(s-1)^n}{s^{n+1}}$$

The solution to the Laguerre equation is

$$y(t) = \mathcal{L}^{-1}\{Y\}$$

$$y_n(t) = \mathcal{L}^{-1}\left\{\frac{(s-1)^n}{s^{n+1}}\right\}$$: Laguerre Polynomial

For $n=0$ $\quad y_0(t) = \mathcal{L}^{-1}\left\{\frac{1}{s}\right\} = 1$

For $n=1$ $\quad y_1(t) = \mathcal{L}^{-1}\left\{\frac{s-1}{s^2}\right\} = \mathcal{L}^{-1}\left\{\frac{1}{s} - \frac{1}{s^2}\right\}$

$$= 1 - t$$

For $n=2$ $\quad y_2(t) = \mathcal{L}^{-1}\left\{\frac{s^2-2s+1}{s^3}\right\} = \mathcal{L}^{-1}\left\{\frac{1}{s} - \frac{2}{s^2} + \frac{1}{s^3}\right\}$

$$y_2(t) = 1 - 2t + \frac{1}{2}t^2$$

Power Series solutions of ODE's

Recall If f is ∞ - differentiable on a certain interval continuing a, we define the Taylor Series of f near $x = a$ is follows:

$$\sum_{n=0}^{\infty} \frac{f^{(n)}(a)}{n!}(x-a)^n = f(a) + \frac{f'(a)}{1!}(x-a) + \frac{f''(a)}{2!}(x-a)^2 + \ldots$$

If $a = 0$, Taylor series is called Mclaurine Series:

$$\sum_{n=0}^{\infty} \frac{f^{(n)}(0)}{n!} x^n$$

These are examples of what we call

Power series:

Power series $\sum_{n=0}^{\infty} C_n (x-a)^n$

Basic Mclaurin Series

① $\frac{1}{1-x} = 1 + x + x^2 + x^3 + \ldots$

$$= \sum_{n=0}^{\infty} x^n \; ; \; -1 < x < 1$$

② $e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \dots = \sum_{n=0}^{\infty} \frac{x^n}{n!}$; all x

③ $\cos x = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \dots = \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n}}{(2n)!}$; all x

④ $\sin x = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \dots = \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n+1}}{(2n+1)!}$; all x

Ex/ Find the first $\underline{4}$ terms in the power series expansion of the function y, solution to the IVP:

$$y'' + y = 0 \ ; \quad y(0) = 1, \ y'(0) = 2$$

Sol $y = \sum_{n=0}^{\infty} a_n x^n = a_0 + a_1 x + a_2 x^2 + a_3 x^3 + \dots$

$$y' = \sum_{n=1}^{\infty} n a_n x^{n-1} = a_1 + 2a_2 x + 3a_3 x^2 + \dots$$

$$y'' = \sum_{n=2}^{\infty} n(n-1) a_n x^{n-2}$$

$y(0) = 1 \Rightarrow a_0 = 1$

$y'(0) = 2 \Rightarrow a_1 = 2$

Back to the ODE:

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$$\underbrace{\sum_{n=2}^{\infty} n(n-1)\,a_n X^{n-2}}_{\text{Let } k=n-2} + \underbrace{\sum_{n=0}^{\infty} a_n X^n}_{n=k} = 0$$

$$\sum_{k=0}^{\infty} (k+2)(k+1)\,a_{k+2} X^k + \sum_{k=0}^{\infty} a_k X^k = 0$$

$$\sum \left((k+1)(k+2)\,a_{k+2} + a_k\right) X^k = 0$$