Lec 22

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Last time Solving IVP'S using Laplace

$\mathcal{L}\{y'\} = s\,\mathcal{L}\{y\} - y(0)$

$\mathcal{L}\{y''\} = s^2\,\mathcal{L}\{y\} - s\,y(0) - y'(0)$

$\mathcal{L}\{y'''\} = s^3\,\mathcal{L}\{y\} - s^2 y(0) - s\,y'(0) - y''(0)$

Ex 2/ Solve the IVP :

$y'' - 6y' + 9y = \sin t + \delta(t-1)\,;\ y(0) = 1\,,\ y'(0) = 10$

Here $\delta(t-1)$ is the Dirac-delta Function at $t=1$

Sol

Let $Y(s) = \mathcal{L}\{\underline{y(t)}\}$

Apply $\mathcal{L}$ to both sides :

$\mathcal{L}\{y''\} - 6\,\mathcal{L}\{y'\} + 9\,\mathcal{L}\{y\} = \mathcal{L}\{\sin t\} + \mathcal{L}\{\delta(t-1)\}$

$\Rightarrow s^2 Y - s\,y(0) - y'(0) - 6\left[sY - y(0)\right] + 9Y = \frac{1}{s^2+1} + e^{-s}$

$\underline{s^2 Y} - s - 10 - \underline{6sY} + 6 + \underline{9Y} = \frac{1}{s^2+1} + e^{-s}$

$(s^2 - 6s + 9)\,Y = s + 4 + \frac{1}{s^2+1} + e^{-s}$

$\Rightarrow Y = \frac{s+4}{s^2-6s+9} + \frac{1}{(s^2+1)(s^2-6s+9)} + \frac{e^{-s}}{s^2-6s+9}$

$Y = \frac{s+4}{(s-3)^2} + \frac{1}{(s^2+1)(s-3)^2} + \frac{e^{-s}}{(s-3)^2}$

So the solution to the IVP is :

$$y(t) = \mathcal{L}^{-1}\{Y(s)\}$$

$$y(t) = \mathcal{L}^{-1}\left\{\frac{s+4}{(s-3)^2}\right\} + \mathcal{L}^{-1}\left\{\frac{1}{(s^2+1)(s-3)^2}\right\} + \mathcal{L}^{-1}\left\{\frac{e^{-s}}{(s-3)^2}\right\}$$

• For $\mathcal{L}^{-1}\left\{\frac{s+4}{(s-3)^2}\right\}$

$$\frac{s+4}{(s-3)^2} = \frac{A}{(s-3)} + \frac{B}{(s-3)^2} = \frac{A(s-3)+B}{(s-3)^2}$$

$\Rightarrow A = 1$

$-3A + B = 4 \Rightarrow B = 7$

So $\frac{s+4}{(s-3)^2} = \frac{1}{s-3} + \frac{7}{(s-3)^2}$

$$\Rightarrow \mathcal{L}^{-1}\left\{\frac{s+4}{(s-3)^2}\right\} = \mathcal{L}^{-1}\left\{\frac{1}{s-3}\right\} + 7\mathcal{L}^{-1}\left\{\frac{1}{(s-3)^2}\right\}$$

$$\boxed{= e^{3t} + 7te^{3t}}$$

• For $\mathcal{L}^{-1}\left\{\frac{1}{(s^2+1)(s-3)^2}\right\}$

$$\frac{1}{(s^2+1)(s-3)^2} = \frac{As+B}{(s^2+1)} + \frac{C}{s-3} + \frac{D}{(s-3)^2}$$

$$= \frac{(As+B)(s-3)^2 + C(s^2+1)(s-3) + D(s^2+1)}{(s^2+1)(s-3)^2}$$

$$= \frac{(As+B)(s^2-6s+9) + C(s^3-3s^2+s-3) + D(s^2+1)}{(s^2+1)(s-3)^2}$$

$$\begin{array}{ll} s^3 & \Rightarrow A + C = 0 \\ s^2 & -6A + B - 3C + D = 0 \\ s & 9A - 6B + C = 0 \\ c & 9B - 3C + D = 1 \end{array} = \left[\begin{array}{cccc|c} 1 & 0 & 1 & 0 & 0 \\ -6 & 1 & -3 & 1 & 0 \\ 9 & -6 & 1 & 0 & 0 \\ 0 & 9 & -3 & 1 & 1 \end{array}\right]$$

$$\left[\begin{array}{cccc|c} 1 & 0 & 0 & 0 & 3/50 \\ 0 & 1 & 0 & 0 & 2/25 \\ 0 & 0 & 1 & 0 & -3/50 \\ 0 & 0 & 0 & 1 & 1/10 \end{array}\right] \quad \begin{array}{l} A = 3/50 \\ B = 2/25 \\ C = -3/50 \\ D = 1/10 \end{array}$$

So $\dfrac{1}{(s^2+1)(s-3)^2} = \dfrac{3}{50}\dfrac{s}{s^2+1} + \dfrac{2}{25}\cdot\dfrac{1}{s^2+1} - \dfrac{3}{50}\dfrac{1}{s-3} + \dfrac{1}{10}\dfrac{1}{(s-3)^2}$

$$\mathcal{L}^{-1}\left\{\frac{1}{(s^2+1)(s-3)^2}\right\} = \frac{3}{50}\mathcal{L}^{-1}\left\{\frac{s}{s^2+1}\right\} + \frac{2}{25}\mathcal{L}^{-1}\left\{\frac{1}{s^2+1}\right\} - \frac{3}{50}\mathcal{L}^{-1}\left\{\frac{1}{s-3}\right\} + \frac{1}{10}\mathcal{L}^{-1}\left\{\frac{1}{(s-3)^2}\right\}$$

$$= \frac{3}{50}\cos t + \frac{2}{25}\sin t - \frac{3}{50}e^{3t} + \frac{1}{10}te^{3t}$$

- For $\mathcal{L}^{-1}\left\{\dfrac{e^{-s}}{(s-3)^2}\right\}$ $\quad F(s) = \dfrac{1}{(s-3)^2}$

$$f(t) = \mathcal{L}^{-1}\{F(s)\} = \mathcal{L}^{-1}\left\{\frac{1}{(s-3)^2}\right\} = te^{3t}$$

$$\mathcal{L}^{-1}\left\{\frac{e^{-s}}{(s-3)^2}\right\} = u(t-1)\cdot F(t-1)$$

$$= \underline{u(t-1)(t-1)e^{3(t-1)}}$$ (Second shifting theorem)

So $y(t) = e^{3t} + 7te^{3t} + \left(\frac{3}{50}\cos t + \frac{2}{25}\sin t - \frac{3}{50}e^{3t} + \frac{1}{10}te^{3t}\right)$
$+ u(t-1)(t-1)e^{3(t-1)}$

$$= \frac{47}{50}e^{3t} + \frac{71}{10}te^{3t} + \frac{3}{50}\cos t + \frac{2}{25}\sin t + u(t-1)(t-1)e^{3t-3}$$

Ex/ Solve the IVP:

$$y'' - y = \begin{cases} 0 & \text{if } 0 \leq t < \pi \\ \cos t & \text{if } t \geq \pi \end{cases}$$

$y(0) = 2$, $y'(0) = 1$

Sol

Note that $r(t) = \begin{cases} 0 & \text{if } 0 \leq t < \pi \\ \cos t & \text{if } \geq \pi \end{cases}$

$$= \cos t \underbrace{\begin{cases} 0 \; ; & 0 \leq t < \pi \\ 1 \; ; & t \geq \pi \end{cases}}_{u(t-\pi)}$$

So $y'' - y = \cos t \; u(t-\pi)$

Let $Y(s) = \mathcal{L}\{y(t)\}$. Apply $\mathcal{L}$ to both sides

$$\mathcal{L}\{y''\} - \mathcal{L}\{y\} = \mathcal{L}\{u(t-\pi)\underbrace{\cos t}_{f(t-\pi)}\}$$

$$f'(t) = \cos(t+\pi)$$
$$= -\cos(t) \quad \text{trig Identity}$$

$$\Rightarrow s^2 \underline{Y} - s\,y(0) - y'(0) - \underline{Y} = = e^{-\pi s} \mathcal{L}\{-\cos(t)\}$$

$$(s^2-1)Y - 2s - 1 = -e^{-\pi s} \frac{s}{s^2+1}$$

$$\text{So } Y = \frac{2s+1}{s^2-1} - e^{-\pi s} \frac{s}{(s^2+1)(s^2-1)}$$

$$y(t) = \mathcal{L}^{-1}\left\{\frac{2s+1}{s^2-1}\right\} - \mathcal{L}^{-1}\left\{e^{-\pi s} \frac{s}{(s^2+1)(s^2-1)}\right\}$$

• For $\mathcal{L}^{-1}\left\{\frac{2s+1}{s^2-1}\right\}$

$$\frac{2s+1}{s^2-1} = \frac{2s+1}{(s-1)(s+1)} = \frac{A}{s-1} + \frac{B}{s+1}$$

$$= \frac{A(s+1) + B(s-1)}{(s-1)(s+1)}$$

$s \Rightarrow A + B = 2 \qquad A = \frac{3}{2}$

$A - B = 1 \qquad B = 2 - \frac{3}{2} = \frac{1}{2}$

$$\text{So } \frac{2s+1}{s^2-1} = \frac{3}{2}\,\frac{1}{s-1} + \frac{1}{2}\,\frac{1}{s+1}$$

$$\mathcal{L}^{-1}\left\{\frac{2s+1}{s^2-1}\right\} = \frac{3}{2}\mathcal{L}^{-1}\left\{\frac{1}{s-1}\right\} + \frac{1}{2}\mathcal{L}^{-1}\left\{\frac{1}{s+1}\right\}$$

$$\boxed{= \frac{3}{2} e^{t} + \frac{1}{2} e^{-t}}$$

• For $\mathcal{L}^{-1}\left\{ e^{-\pi s} \dfrac{s}{(s^2+1)(s^2-1)} \right\}$

$$F(s) = \frac{s}{(s^2+1)(s^2-1)} = \frac{s}{(s^2+1)(s-1)(s+1)}$$

$$\frac{s}{(s^2+1)(s-1)(s+1)} = \frac{As+B}{s^2+1} + \frac{C}{s-1} + \frac{D}{s+1} \quad (*)$$

$$= \frac{(As+B)(s-1)(s+1) + C(s^2+1)(s+1) + D(s^2+1)(s-1)}{(s^2+1)(s-1)(s+1)}$$

$$= \frac{As^3 - As + Bs^2 - B + Cs^3 + Cs^2 + Cs + C + Ds^3 - Ds^2 + Ds - D}{(s^2+1)(s-1)(s+1)}$$

$$\left.\begin{array}{ll} s^3: & A + C + D = 0 \\ s^2: & B + C - D = 0 \\ s: & -A + C + D = 1 \\ c: & -B + C - D = 0 \end{array}\right\} \left[\begin{array}{cccc|c} 1 & 0 & 1 & 1 & 0 \\ 0 & 1 & 1 & -1 & 0 \\ -1 & 0 & 1 & 1 & 1 \\ 0 & -1 & 1 & -1 & 0 \end{array}\right]$$

$$\left[\begin{array}{cccc|c} 1 & 0 & 0 & 0 & -1/2 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 1/4 \\ 0 & 0 & 0 & 1 & 1/4 \end{array}\right] \quad \begin{array}{l} A = -1/2 \\ B = 0 \\ C = 1/4 \\ D = 1/4 \end{array}$$

Back to $(*)$:

$$\frac{s}{(s^2+1)(s^2-1)} = -\frac{1}{2}\frac{s}{s^2+1} + \frac{1}{4}\cdot\frac{1}{s-1} + \frac{1}{4}\frac{1}{s+1}$$

$$f(t) = \mathcal{L}^{-1}\left\{\frac{s}{(s^2+1)(s^2-1)}\right\} = -\frac{1}{2}\mathcal{L}^{-1}\left\{\frac{s}{s^2+1}\right\} + \frac{1}{4}\mathcal{L}^{-1}\left\{\frac{1}{s-1}\right\} + \frac{1}{4}\mathcal{L}^{-1}\left\{\frac{1}{s+1}\right\}$$

$$= -\frac{1}{2}\cos t + \frac{1}{4}e^{t} + \frac{1}{4}e^{-t}$$

$$= -\frac{1}{2}\cos t + \frac{1}{4}e^{t} + \frac{1}{4}e^{-t}$$

So $$y(t) = \frac{3}{2}e^{t} + \frac{1}{2}e^{-t} - \left[u(t-\pi)\underbrace{\left(-\frac{1}{2}\cos(t-\pi) + \frac{1}{4}e^{(t-\pi)} + \frac{1}{4}e^{-(t-\pi)}\right)}_{f(t-\pi)}\right]$$

$$y(t) = \frac{3}{2}e^{t} + \frac{1}{2}e^{-t} - \left[u(t-\pi)\left(\frac{1}{2}\cos(t) + \frac{1}{4}e^{(t-\pi)} + \frac{1}{4}e^{(\pi-t)}\right)\right]$$

Laplace of $\frac{f(t)}{t}$

Theorem Let $f(t)$ be a function such that $\mathcal{L}\{f(t)\}$ and $\mathcal{L}\left\{\frac{f(t)}{t}\right\}$ exist. If $\lim_{t\to 0}\frac{f(t)}{t}$ exists, then

$$\boxed{\mathcal{L}\left\{\frac{f(t)}{t}\right\} = \int_{s}^{+\infty} F(x)\,dx}$$ where $F(s) = \mathcal{L}\{f(t)\}$

Ex Find $\mathcal{L}\left\{\frac{e^{t}-1}{t}\right\}$

Sol $$\lim_{t\to 0}\frac{f(t)}{t} = \lim_{t\to 0}\frac{e^{t}-1}{t} \overset{\text{L'H's}}{=} \frac{e^{t}}{1} = 1 \text{ exists}$$

So $$\mathcal{L}\{e^{t}-1\} = \frac{1}{s-1} - \frac{1}{s}$$

$+\infty$

$$\int_5^{\infty} \frac{1}{x-1} - \frac{1}{x}\, dx = \lim_{L \to \infty} \int_5^{L} \left(\frac{1}{x-1} - \frac{1}{x} \right) dx$$