Lec 21 Convolution; Laplace as a Tool to Solving IVP's

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ODE Lec 21

Last time

① first shifting theorem

② Second shifting theorem

③ Multiplication by t^n Rule

Ex/ Find $\mathcal{L}\{t\,\underbrace{e^{-2t}\sin(3t)}_{F(x)}\}$

$$F(s) = \mathcal{L}\{e^{-2t}\sin(3t)\} = \frac{3}{(s+2)^2+9}.$$ First shifting theorem

$$= \frac{3}{s^2+4s+13} \Rightarrow \mathcal{L}\{te^{-2t}\sin(3t)\}$$

$$= (-1)^1 F'(s)$$

$$= -\left(\frac{3(2s+4)}{(s^2+4s+13)^2}\right)$$

Ex/ Find $\mathcal{L}\{t^2\,\underbrace{u(t-1)e^{2t}}_{F(t)}\}$

$$F(s) = \mathcal{L}\{u(t-1)e^{2t}\} =$$

$$f(t-1) = e^{2t} \Rightarrow f(t) = e^{2(t+1)} = e^{2}\cdot e^{2t}$$

$F(s) = \mathcal{L}\{f(t)\} = \mathcal{L}\{e^2 e^{2t}\} = e^{-s} e^2 \mathcal{L}\{e^{2t}\} = \frac{e^2}{s-2} e^{-s}$

So $\mathcal{L}\{t^2 u(t-1) e^{2t}\} = (-1)^2 F''(s)$

$F(s) = \frac{e^2 \cdot e^{-s}}{s-2} \Rightarrow F'(s) = \frac{-e^{2-s}(s-2) - e^{2-s}}{(s-2)^2} = \frac{-e^{2-s}(s-1)}{(s-2)^2}$

$F''(s) = \frac{\left[e^{2-s}(s-1) - e^{2-s}\right](s-2)^2 + 2(s-2)e^{2-s}(s-1)}{(s-2)^4}$

$= \frac{e^{2-s}(s-2)\cancel{(s-2)^2} + 2\cancel{(s-2)}e^{2-s}(s-1)}{\cancel{(s-2)^4}}$

$= \frac{e^{2-s}(s-2)^2 + 2(s-1)e^{2-s}}{(s-1)^3}$

$\mathcal{L}\{t^2 u(t-1) e^{2t}\}$

Convolution

We know that $\mathcal{L}\{f(t)+g(t)\} = \mathcal{L}\{f(t)\} + \mathcal{L}\{g(t)\}$

The result is not true if "+" is replaced with "*"

$$\mathcal{L}\{f(t)\cdot g(t)\} \neq \mathcal{L}\{f(t)\}\cdot\mathcal{L}\{g(t)\}$$

However, the result is true if we redefine the way we multiply functions!

<u>Definition</u> Given 2 functions $f(t)$ and $g(t)$ (piecewise continuous and of exponential order), the <u>convulation</u> of f and g is the function $f*g$, defined by

$$\boxed{(f*g)(t) = \int_0^t f(x)g(t-x)\,dx}$$

Properties: ① $f * g = g * f$

② $f * (g * h) = (f * g) * h$

③ $f * (g + h) = f * g + f * h$

④ $f * 0 = 0$

Ex 1 $f(t) = t$, $g(t) = t^2$. Find $f * g$:

$$t * t^2 = \int_0^t (t - x)\, x^2\, dx = \int_0^t t x^2 - x^3\, dx = \frac{t x^3}{3} - \frac{x^4}{4} \Big|_0^t$$

$$= \frac{t^4}{3} - \frac{t^4}{4} = \boxed{\frac{1}{12} t^4}$$

Ex 2/ $f(t) = e^{-t}$, $g(t) = e^{3t}$

Find $f * g$

Sol/ $$e^{-t} * e^{3t} = \int_0^t e^{-(t-x)} * e^{3x}\, dx$$

$$\int_0^t e^{-t} e^{4x}\, dx = e^{-t} \left[\frac{1}{4} e^{4x} \right]_0^t = e^{-t} \left[\frac{1}{4} e^{4t} - \frac{1}{4} \right]$$

$$= \frac{1}{4} e^{3t} - \frac{1}{4} e^{-t} = \frac{1}{4} \left(e^{3t} - e^{-t} \right)$$

$$e^{-t} * e^{3t} = \frac{1}{4} e^{3t} - \frac{1}{4} e^{-t}$$

Theorem (convolution theorem) Let $f(t), g(t)$ 2 functions of exponential order α and piecewise continuous on

$$[0, +\infty[$$

Let $F(s) = \mathcal{L}\{f(t)\}$, $G(s) = \mathcal{L}\{g(t)\}$. Then

$$\boxed{\begin{array}{l} \mathcal{L}\{f(t) * g(t)\} = F(s) \cdot G(s) \quad \text{or} \\ \mathcal{L}^{-1}\{F(s) * G(s)\} = f(t) * g(t) \end{array}}$$

Ex 1/ Find $\mathcal{L}^{-1}\left\{\frac{1}{s^2 - 4s + 3}\right\}$

So $\mathcal{L}^{-1}\left\{\frac{1}{s^2-4s+3}\right\} = \mathcal{L}^{-1}\left\{\frac{1}{(s-1)(s-3)}\right\}$

$$= \mathcal{L}^{-1}\left\{\frac{1}{s-1} \cdot \frac{1}{s-3}\right\} = \mathcal{L}^{-1}\left\{\frac{1}{s-1}\right\} * \mathcal{L}^{-1}\left\{\frac{1}{s-3}\right\}$$

$$= e^{t} * e^{3t} = \int_0^t e^{(t-x)} e^{3x}\, dx = \int_0^t e^{t} \cdot e^{2x}\, dx$$

$$= e^{t} \int_0^t e^{2x}\, dx = e^{t}\left[\frac{1}{2} e^{2x}\right]_0^t = e^{t}\left[\frac{1}{2} e^{2t} - \frac{1}{2}\right]$$

$$\boxed{= \frac{1}{2} e^{3t} - \frac{1}{2} e^{t}}$$

Ex/ Find $\mathcal{L}\{u(t-1)(t^2+2) * \sin(3t)\}$

Sol/ $\mathcal{L}\{u(t-)(t^2+2) * \sin(3t)\} =$

$$\mathcal{L}\{u(t-1)\underbrace{(t^2+2)}_{f(t-1)}\} \cdot \mathcal{L}\{\sin(3t)$$

$$f(t) = (t+1)^2 + 2 = t^2 + 2t + 3$$

$$\mathcal{L}\{u(t-1)(t^2+2)\} = e^{-s}\,\mathcal{L}\{t^2+2t+3\}$$ second shifting theorem

$$= e^{-s}\left(\frac{2}{s^3} + \frac{2}{s^2} + \frac{3}{s}\right)$$

So $\mathcal{L}\{u(t-1)(t^2+2) * \sin(3t)\} = e^{-s}\left(\frac{2}{s^3} + \frac{2}{s^2} + \frac{3}{s}\right) \cdot \frac{3}{s^2+9}$

Ex/ Find $\mathcal{L}^{-1}\left\{\frac{1}{(s^2+1)^2}\right\}$

use: $\sin(a-b) = \sin a \cos b - \cos a \sin b$

$\cos(2\alpha) = 2\cos^2\alpha - 1 = 1 - 2\sin^2\alpha$

$\sin(2\alpha) = 2\sin\alpha\cos\alpha$

Sol/ $\mathcal{L}^{-1}\left\{\frac{1}{(s^2+1)^2}\right\} = \mathcal{L}^{-1}\left\{\frac{1}{s^2+1}\cdot\frac{1}{s^2+1}\right\}$

$= \mathcal{L}^{-1}\left\{\frac{1}{s^2+1}\right\} * \mathcal{L}^{-1}\left\{\frac{1}{s^2+1}\right\} = \sin t * \sin t$

$= \int_0^t \sin x \sin(t-x)\,dx$

$= \int_0^t \sin x(\sin t\cos x - \cos t \sin x)\,dx$

$= \sin t \int_0^t \sin x \cos x\,dx - \cos t \int_0^t \sin^2 x\,dx$

$= \frac{\sin t}{2}\int_0^t \sin(2x)\,dx - \cos t \int_0^t \frac{1-\cos(2x)}{2}\,dx$

$= \frac{\sin t}{2}\left[-\frac{1}{2}\cos(2x)\right]_0^t - \frac{\cos t}{2}\left[x - \frac{1}{2}\sin(2x)\right]_0^t$

$= \frac{1}{2}\sin t\left[-\frac{1}{2}\cos(2x) + \frac{1}{2}\right] - \frac{1}{2}\cos t\left[t - \frac{1}{2}\sin(2t)\right]$

<u>La place as a tool to solve IVP's</u>

If $Y(s) = \mathcal{L}\{y(t)\}$, what is $\mathcal{L}\{y'(t)\}$?

$\mathcal{L}\{y'(t)\} = \int_0^{+\infty} e^{-st} y'(t)\,dt =$

$$\lim_{L \to +\infty} \int_0^L \underbrace{e^{-st} y'(t)\,dt}_{\text{by Parts}}$$

$$u = e^{-st} \qquad v' = y'$$
$$u' = -s e^{-st} \qquad v = y$$

$$\int e^{-st} y'(t)\,dt = y e^{-st} + s \int e^{-st} y(t)\,dt$$

$$\text{So } \int_0^L e^{-st} y'(t)\,dt = \left[y(t) e^{-st} + s \int e^{-st} y(t)\,dt \right]_0^L$$

$$= y(L) e^{-sL} + s \int_0^L e^{-st} y(t)\,dt - y(0)$$

$$\mathcal{L}\{y'(t)\} = \lim_{L \to +\infty} \left[y(L) \overset{\nearrow 0}{e^{-sL}} + s \int_0^L e^{-st} y(t)\,dt - y(0) \right]$$

$$\boxed{\mathcal{L}\{y'(t)\} = s\,\mathcal{L}\{y(t)\} - y(0)}$$

$$\text{So } \mathcal{L}\{y''(t)\} = s\,\mathcal{L}\{y'(t)\} - y'(0)$$

$$= s\left[s\,\mathcal{L}\{y\} - y(0) \right) - y'(0) \longrightarrow$$

$$\mathcal{L}\{y''\} = s^2\mathcal{L}\{y\} - s\,y(0) - y'(0)$$

$$\mathcal{L}\{y'(t)\} = s\,\mathcal{L}\{y(t)\} - y(0)$$

$$\mathcal{L}\{y''(t)\} = s^2\,\mathcal{L}\{y(t)\} - s\,y(0) - y'(0)$$

$$\mathcal{L}\{y''(t)\} = s^3\,\mathcal{L}\{y(t)\} - s^2 y(0) - s\,y'(0) - y''(0)$$

In General:

$$\mathcal{L}\{y^{(n)}(t)\} = s^n\,\mathcal{L}\{y(t)\} - s^{n-1}y(0) \dots - y^{(n-1)}(0)$$

Now we can solve IVP's, using Laplace:

1. Let $Y(s) = \mathcal{L}\{y(t)\}$
2. Apply Laplace to both sides of the ODE
3. Isolate $Y(s)$
4. The solution is $y(t) = \mathcal{L}^{-1}\{Y(s)\}$

Ex 1. s

Ex 1 Solve the IVP: $y'' + 4y' - 5y = 3e^t$, $y(0) = 1$; $y'(0) = 0$

Sol $Y(s) = \mathcal{L}\{y(t)\}$ Apply $\mathcal{L}$ to bothsides:

$\mathcal{L}\{y''\} + 4\mathcal{L}\{y'\} - 5\mathcal{L}\{y\} = 3\mathcal{L}\{e^t\}$

$s^2 Y - s y(0) - y'(0) + 4[sY - y(0)] - 5Y = \frac{3}{s-1}$

$s^2 Y - s - 0 + 4sY - 4 - 5Y = \frac{3}{s-1}$

$(s^2 + 4 - 5)Y = s + 4 + \frac{3}{s-1}$

$Y = \frac{s+4}{s^2+4s-5} + \frac{3}{(s-1)(s^2+4s-5)}$

The solution is $y(t) = \mathcal{L}^{-1}\{Y(s)\}$

$y(t) = \mathcal{L}^{-1}\left\{\frac{s+4}{s^2+4s-5}\right\} + \mathcal{L}^{-1}\left\{\frac{3}{(s-1)(s^2+4s-5)}\right\}$ (*)

- $\frac{s+4}{s^2+4s-5} = \frac{s+4}{(s-1)(s+5)} = \frac{A}{s-1} + \frac{B}{s+5} = \frac{(A+B)s + 5A - B}{(s-1)(s+5)}$

$\Rightarrow A + B = 1$ ① ① + ② $= 6A = 5 \Rightarrow A = \frac{5}{6}$

$5A - B = 4$ ② $\frac{5}{6} + B = 1 \Rightarrow B = \frac{1}{6}$

$\frac{s+4}{s^2+4s-5} = \frac{5}{6}\frac{1}{s-1} + \frac{1}{6}\frac{1}{s+5} \Rightarrow \mathcal{L}^{-1}\left\{\frac{s+4}{s^2+4s-5}\right\}$

$$= \frac{5}{6}\mathcal{L}^{-1}\left\{\frac{1}{s-1}\right\} + \frac{1}{6}\mathcal{L}^{-1}\left\{\frac{1}{s+5}\right\} = \frac{5}{6}e^{t} + \frac{1}{6}e^{-5t}$$

$$\boxed{\mathcal{L}^{-1}\left\{\frac{s+4}{s^2+4s-5}\right\} = \frac{5}{6}e^{t} + \frac{1}{6}e^{-5t}}$$

- $$\frac{3}{(s-1)\underbrace{(s^2+4s-5)}_{(s-1)(s+5)}} = \frac{3}{(s-1)^2(s+5)} = \frac{A}{s-1} + \frac{B}{(s-1)^2} + \frac{C}{s+5}$$

$$= \frac{A(s-1)(s+5) + B(s+5) + C(s-1)^2}{(s-1)^2(s+5)}$$

$$= A + C = 0$$

$$4A + B - 2C = 0$$

$$-5A + 5B + C = 3$$

$$\left[\begin{array}{ccc|c} 1 & 0 & 1 & 0 \\ 4 & 1 & -2 & 0 \\ -5 & 5 & 1 & 3 \end{array}\right] \sim \left[\begin{array}{ccc|c} 1 & 0 & 0 & -1/12 \\ 0 & 1 & 0 & 1/2 \\ 0 & 0 & 1 & 1/12 \end{array}\right]$$

$A = -\frac{1}{12}$ $\quad B = \frac{1}{2}$ $\quad C = \frac{1}{12}$