Lec 20
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Ex Find $\mathcal{L}^{-1}\left\{\frac{s}{s^2+4s+5}\right\}$

$$\mathcal{L}^{-1}\left\{\frac{s}{s^2+4s+5}\right\} = \mathcal{L}^{-1}\left\{\frac{s}{s^2+4s+4-4+5}\right\} = \mathcal{L}^{-1}\left\{\frac{s}{(s+2)^2+1}\right\} =$$

$$= \mathcal{L}^{-1}\left\{\frac{s+2-2}{(s+2)^2+1}\right\} = \mathcal{L}^{-1}\left\{\frac{s+2}{(s+2)^2+1}\right\} - 2\mathcal{L}^{-1}\left\{\frac{1}{(s+2)^2+1}\right\}$$

$$= e^{-2t}\cos t - 2e^{-2t}\sin t$$

Ex Find $\mathcal{L}^{-1}\left\{\frac{3s+2}{(s+1)^3(s^2+2s+2)}\right\}$

Partial fraction

$$\frac{3s+2}{(s+1)^3(s^2+2s+2)} = \frac{A}{s+1} + \frac{B}{(s+1)^2} + \frac{C}{(s+1)^3} + \frac{Ds+E}{s^2+2s+2}$$

$$= \frac{A(s+1)^2(s^2+2s+2) + B(s+1) + C(s^2+2s+2) + (Ds+E)(s+1)^3}{(s+1)^3(s^2+2s+2)}$$

s^4: $A + D = 0$

s^3: $4A + B + 3D + E = 0$

s^2: $7A + 3B + C + 3D + 3E = 0$

s: $6A + 4B + 2C + D + 3E = 3$

c: $2A + 2B + 2C + E = 2$

$$\left[\begin{array}{ccccc|c} 1 & 0 & 0 & 1 & 0 & 0 \\ 4 & 1 & 0 & 3 & 1 & 0 \\ 7 & 3 & 1 & 3 & 3 & 0 \\ 6 & 4 & 2 & 1 & 3 & 3 \\ 2 & 2 & 2 & 6 & 1 & 2 \end{array}\right] \sim$$

$$\left[\begin{array}{ccccc|c} 1 & 0 & 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 & 0 & 3 \\ 0 & 0 & 1 & 0 & 0 & -1 \\ 0 & 0 & 0 & 1 & 0 & -1 \\ 0 & 0 & 0 & 0 & 1 & 4 \end{array}\right]$$

$A = 1$

$B = 3$

$C = -1$

$D = -1$

$E = 4$

(*) gives

$$\frac{3s+2}{(s+1)^3(s^2+2s+2)} = \frac{1}{s+1} + \frac{3}{(s+1)^2} - \frac{1}{(s+1)^3} - \frac{s-4}{\underbrace{s^2+2s+2}_{(s+1)^2+1}}$$

complete the square

$$= \frac{1}{s+1} + \frac{3}{(s+1)^2} - \frac{1}{(s+1)^3} - \left(\frac{s+1-5}{(s+1)^2+1}\right)$$

$$= \frac{1}{s+1} + \frac{3}{(s+1)^2} - \frac{1}{(s+1)^3} - \frac{s+1}{(s+1)^2+1} + \frac{5}{(s+1)^2+1}$$

$$= \mathcal{L}^{-1}\left\{\frac{1}{s+1}\right\} + 3\mathcal{L}^{-1}\left\{\frac{1}{(s+1)^2}\right\} - \mathcal{L}^{-1}\left\{\frac{1}{(s+1)^3}\right\} - \mathcal{L}^{-1}\left\{\frac{s+1}{(s+1)^2+1}\right\} + 5\mathcal{L}^{-1}\left\{\frac{1}{(s+1)^2+1}\right\}$$

$$= e^{-t} + 3te^{-t} - \frac{1}{2}t^2e^{-t} - e^{-t}\cos t + 5e^{-t}\sin t$$

The second shifting theorem

Let $a > 0$, $F(s) = \mathcal{L}\{f(t)\}$, then

$$\mathcal{L}\{u(t-a)f(t-a)\} = e^{-as}F(s) \text{ or}$$

$$\mathcal{L}\{u(t-a)f(t-a)\} = e^{-as} F(s) \text{ or}$$

$$\boxed{\mathcal{L}^{-1}\{e^{-as}F(s)\} = u(t-a)f(t-a)}$$

Ex/ Find $\mathcal{L}\{u(t-2)\underbrace{(t^2-3t+2)}_{f(t-2)}\}$

Sol $f(t-2) = t^2-3t+2$

$$f(t) = f(t-2+2) = (t+2)^2 - 3(t+2)+2$$

$$= t^2+t$$

By second shifting theorem:

$$\mathcal{L}\{u(t-2)(t^2-3t+2)\} = e^{-2s}\mathcal{L}\{f(t)\}$$

$$= e^{-2s}\mathcal{L}\{t^2+t\} = \boxed{e^{-2s}\left(\frac{2}{s^3}+\frac{1}{s^2}\right)}$$

Ex2/ $\mathcal{L}\{u(t-\pi)\underbrace{\cos(2t)}_{f(t-\pi)}\}$

$$f(t-\pi) = \cos(2t) \Rightarrow \cos(2(t+\pi))$$

$$= \cos(2t+2\pi) = \cos(2t)$$

By the second shifting theorem:

$$\mathcal{L}\{u(t-\pi)\cos(2t)\} = e^{-\pi s}\mathcal{L}\{f(t)\}$$

$$= e^{-\pi s}\mathcal{L}\{\cos(2t)\}$$

$$= e^{-\pi s}\frac{s}{s^2+4}$$

Ex/ Find $\mathcal{L}^{-1}\left\{e^{-3s}\left(\frac{1}{s^3}+\frac{2}{s^2}-\frac{1}{s}\right)\right\}$

Ex/ Find $\mathcal{L}^{-1}\left\{e^{-3s}\underbrace{\left(\frac{1}{s^3}+\frac{2}{s^2}-\frac{1}{s}\right)}_{F(s)}\right\}$

Sol $f(t)=\mathcal{L}^{-1}\left\{\frac{1}{s^3}+\frac{2}{s^2}-\frac{1}{s}\right\}$

$$=\frac{1}{2}t^2+2t-1$$

$$\mathcal{L}^{-1}\left\{e^{-3s}\left(\frac{1}{s^3}+\frac{2}{s^2}-\frac{1}{s}\right)\right\}=u(t-3)f(t-3)$$

$$=u(t-3)\left[\frac{1}{2}(t-3)^2+2(t-3)-1\right]$$

Ex/ $\mathcal{L}^{-1}\left\{\underbrace{\frac{2s}{s^2+9}}_{F(s)}e^{-4s}\right\}$

Sol

$$F(s)=\frac{2s}{s^2+9}$$

$$f(t)=\mathcal{L}^{-1}\left\{\frac{2s}{s^2+9}\right\}=2\mathcal{L}^{-1}\left\{\frac{s}{s^2+9}\right\}=2\cos(3t)$$

$$=u(t-4)\underbrace{2\cos(3(t-4))}_{f(t-4)}$$

Ex/ Find $\mathcal{L}^{-1}\left\{e^{-2s}\underbrace{\frac{s+1}{s^2-5s+6}}_{F(s)}\right\}$

Sol $f(t)=\mathcal{L}^{-1}\left\{\frac{s+1}{s^2-5s+6}\right\}=\mathcal{L}^{-1}\left\{\frac{s+1}{(s-2)(s-3)}\right\}$

Partial Fractions : $\frac{s+1}{\quad}=\frac{A}{\quad}+\frac{B}{\quad}=\frac{A(s-3)+B(s-2)}{\quad}$

Partial Fractions : $\frac{s+1}{(s-2)(s-3)} = \frac{A}{s-2} + \frac{B}{s-3} = \frac{A(s-3)+B(s-2)}{(s-2)(s-3)}$

$$\begin{cases} A+B = 1 & ① \\ -3A-2B = 1 & ② \end{cases}$$

$3① + ② \Rightarrow B = 4$

$① \Rightarrow A - 3$

So $\frac{s+1}{s^2-5s+6} = \frac{-3}{(s-2)} + \frac{4}{(s-3)}$

$$f(t) = \mathcal{L}^{-1}\left\{\frac{s+1}{s^2-5s+6}\right\} = -3e^{2t} + 4e^{3t}$$

By the second shifting theorem :

$$\mathcal{L}^{-1}\left\{e^{-2s}\frac{s+1}{s^2-5s+6}\right\} = u(t-2)\left[-3e^{2(t-2)} + 4e^{3(t-2)}\right]$$

Ex/ Find $\mathcal{L}\{u(t-5)\underbrace{e^{4t}}_{f(t-5)}\}$

Sol/

$f(t-5) = e^{4t}$

$f(t) = e^{4(t+5)} = e^{4t+20}$

by second shifting theorem

$$\mathcal{L}\{u(t-5)e^{4t}\} = e^{-5s}\mathcal{L}\{f(t)\}$$

$$= e^{-5s}\mathcal{L}\{e^{4t+20}\}$$

$$= e^{-5s}\mathcal{L}\{e^{4t}\cdot e^{20}\}$$

$$= e^{-5s}\cdot e^{20}\mathcal{L}\{e^{4t}\}$$

$$= \frac{e^{-5s+20}}{(s-4)}$$

Ex/ Find $\mathcal{L}^{-1}\left\{\frac{e^{-s}\, s}{s^2+8s+17}\right\}$

Theorem: (Multiplication by t^n Rule)

If $F(s)=\mathcal{L}\{f(t)\}$, then

$$\mathcal{L}\{t^n f(t)\}=(-1)^n F^{(n)}(s) \quad \text{or} \quad \mathcal{L}^{-1}\{(-1)^n F^{(n)}(s)\}=t^n f(t)$$

Ex/ $\mathcal{L}\{t\sin(2t)\}$

Sol/ $F(s)=\mathcal{L}\{\sin(2t)\}=\frac{2}{s^2+4}$

$$\mathcal{L}\{t\sin(2t)\}=(-1)^1 F'(s)$$

$$=-\frac{0(s^2+4)-(2s)(2)}{(s^2+4)^2}=\frac{4s}{(s^2+4)^2}$$

Ex/ Find $\mathcal{L}\{t\,\underbrace{e^{2t}\cos(3t)}\}$

Sol $F(s)=\mathcal{L}\{e^{2t}\cos(3t)\}=\frac{s-2}{(s-2)^2+9}$ by First shifting theorem

By the multiplication by t^n Rule:

$$\mathcal{L}\{te^{2t}\cos(3t)\}=(-1)^1F'(s)=-\frac{1(s^2-4s+13)-(2s-4)(s-2)}{(s^2-4s+13)^2}$$

$$= \frac{s^2 - 4s - 5}{(s^2 - 4s + 13)^2}$$