Lec 19

November 16, 2018 9:53 AM

Last time Laplace transform

$$\mathcal{L}\{f(t)\} = \int_0^{+\infty} e^{-st} f(t)\,dt$$

① $\mathcal{L}\{1\} = \frac{1}{s} \Longleftrightarrow \mathcal{L}^{-1}\{\frac{1}{s}\} = 1, \; s>0$

② $\mathcal{L}\{t\} = \frac{1}{s^2} \Longleftrightarrow \mathcal{L}\{\frac{1}{s^2}\} = t, \; s>0$

③ $\mathcal{L}\{t^n\} = \frac{n!}{s^{n+1}} \Longleftrightarrow \mathcal{L}^{-1}\{\frac{n!}{s^{n+1}}\} = t^n, \; s>0$

④ $\mathcal{L}\{e^{at}\} = \frac{1}{s-a} \Longleftrightarrow \mathcal{L}^{-1}\{\frac{1}{s-a}\} = e^{at}, \; s>0$

⑤ $\mathcal{L}\{\cos(at)\} = \frac{s}{s^2+a^2} \Longleftrightarrow \mathcal{L}^{-1}\{\frac{s}{s^2+a^2}\} = \cos(at)$

⑥ $\mathcal{L}\{\sin(at)\} = \frac{a}{s^2+a^2} \Longleftrightarrow \mathcal{L}^{-1}\{\frac{a}{s^2+a^2}\} = \sin(at)$

Ex/ Find $\mathcal{L}^{-1}\left\{\frac{2s+3}{s^3-s^2+4s-4}\right\}$

Sol $\frac{2s+3}{s^3-s^2+4s-4} = \frac{2s+3}{s^2(s-1)+4(s-1)} = \frac{2s+3}{(s-1)(s^2+4)}$

$= \frac{A}{s-1} + \frac{Bs+C}{s^2+4}$ *

$= \frac{As^2+4A+Bs^2-Bs+Cs-C}{(s-1)(s^2+4)} = \frac{(A+B)s^2+(-B+C)s+4A-C}{(s-1)(s^2+4)}$

$= \frac{2s+3}{(s-1)(s^2+4)}$

$$\left.\begin{aligned} s^2:&\quad A+B=0 \\ s:&\quad -B+C=2 \\ &\quad 4A-C=3 \end{aligned}\right\}\begin{matrix} ① \\ ② \\ ③ \end{matrix}$$

② + ③ $\Rightarrow 4A - B = 5$ ④

① + ④ $\Rightarrow 5A = 5 \Rightarrow A = 1$

$B = -1$

$C = 1$

$* \Rightarrow \dfrac{2s+3}{s^3-s^2+4s-4} = \dfrac{1}{s-1} + \dfrac{-s+1}{s^2+4} = \dfrac{1}{s-1} - \dfrac{s}{s^2+4} + \dfrac{1}{s^2+4}$

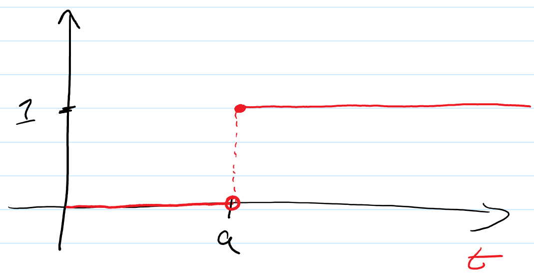
$$\mathcal{L}^{-1}\left\{\frac{2s+3}{s^3-s^2+4s-4}\right\} = \mathcal{L}^{-1}\left\{\frac{1}{s-1}\right\} - \mathcal{L}^{-1}\left\{\frac{s}{s^2+4}\right\} + \mathcal{L}^{-1}\left\{\frac{1}{s^2+4}\right\} =$$

$$e^t - \cos(2t) + \frac{1}{2}\sin(2t)$$

⑦ The heaviside function, also known as the unit step function

Let $a \geq 0$, we define the heaviside function, noted as $u(t-a)$ as follows

$$u(t-a) = \begin{cases} 0 & \text{if } 0 \leq t \leq a \\ 1 & \text{if } t \geq a \end{cases}$$



What is $\mathcal{L}\{u(t-a)\}$?

$$\mathcal{L}\{u(t-a)\} = \int_0^{+\infty} e^{-st} u(t-a)\,dt = \int_0^a e^{-st} \underbrace{u(t-a)}_{0}\,dt + \int_a^{+\infty} e^{-st} \underbrace{u(t-a)}_{1}\,dt$$

$= \int^{+\infty} e^{-st}$ …

$$= \int_a^{+\infty} e^{-st} u(t-a)\, dt = \int_a^{L} e^{-st} u(t-a) dt$$

$$= \lim_{L \to \infty} \int_a^{L} e^{-st} u(t-a) dt$$

$$= \lim_{L \to \infty} \left[\frac{-e^{-st}}{s} \right]_a^L = \lim_{L \to +\infty} \left(\frac{-e^{-sL}}{s} + \frac{e^{-sa}}{s} \right) = \frac{e^{-sa}}{s}$$

⑦ $\mathcal{L}\{u(t-a)\} = \frac{e^{-as}}{s}$

Ex/ $\mathcal{L}\{u(t-3)\} = \frac{e^{-3s}}{s}$

Ex/ $\mathcal{L}^{-1}\left\{\frac{1}{s^2+3} - \frac{e^{-3s}}{s}\right\}$

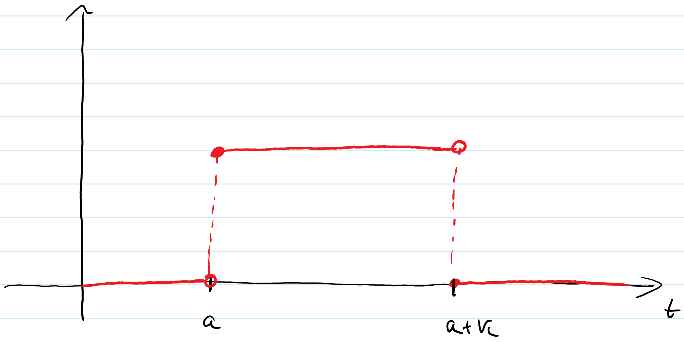
Sol $\mathcal{L}^{-1}\left\{\frac{1}{s^2+3} - \frac{e^{-3s}}{3}\right\} = \underbrace{\mathcal{L}^{-1}\left\{\frac{1}{s^2+3}\right\}}_{\frac{1}{\sqrt{3}}\sin(\sqrt{3}t)} - \underbrace{\mathcal{L}^{-1}\left\{\frac{e^{-3s}}{s}\right\}}_{u(t-3)}$

$\frac{1}{\sqrt{3}} \sin(\sqrt{3}t) \quad - \quad u(t-3)$

⑧ The Drac delta function

Let $K, a > 0$. Define the function $f_{K,a}(t)$ as follows:

$$f_{k,a}(t) = \begin{cases} \frac{1}{k} & \text{if } a \le t < a+k \\ 0 & \text{every where else} \end{cases}$$



It can be shown that $\boxed{f_{k,a}(t) = \frac{1}{k}\left[u(t-a+k) - u(t-a)\right]}$

Definition: The Dirac delta function, noted as $\delta(t-a)$, at $t=a$ is

$$\boxed{\delta(t-a) = \lim_{k \to 0} f_{k,a}(t)}$$

$$\delta(t-a) = \begin{cases} \infty & t = a \\ 0 & t \neq a \end{cases}$$

This represents a force with a huge magnitude in a short period of time!

What is $\mathcal{L}\{\delta(t-a)\}$:

$$\mathcal{L}\{\delta(t-a)\} = \int_0^{+\infty} e^{-st} \underbrace{\delta(t-a)}\,dt = \int_0^{+\infty} e^{-st}\left(\lim_{k\to 0} f_{k,a}(t)\right)dt =$$

$$\lim_{k\to 0}\int_0^{+\infty} e^{-st} f_{k,a}(t)\,dt = \lim_{k\to 0}\int_0^{+\infty} e^{-st}\frac{1}{k}\left[u(t-a) - u(t-(a+k))\right]dt$$

$$= \lim_{k\to 0}\mathcal{L}\left\{\frac{1}{k}\left[u(t-a) - u(t-(a+k))\right]\right\}$$

$$= \mathcal{L}\{\delta(t-a)\} = \lim_{k\to 0}\frac{1}{k}\left[\frac{e^{-as}}{s} - \frac{e^{-(a+k)s}}{s}\right] =$$

$$= \lim_{k\to 0}\frac{1}{k}\left[\frac{e^{-as}}{s} - \frac{e^{-as}\cdot e^{-ks}}{s}\right] = \lim_{k\to 0}\frac{e^{-as}}{sk}\left[1 - e^{-ks}\right]$$

$$= \lim_{k\to 0}\frac{e^{-as}}{s}\left[\frac{1-e^{-ks}}{k}\right] = \lim_{k\to 0}\frac{e^{-as}}{s}\left(\frac{se^{-ks}}{1}\right) \quad \text{L'H}$$

$$= e^{-as}$$

$$\mathcal{L}\{\delta(t-a)\} = e^{-as} \iff \mathcal{L}^{-1}\{e^{-as}\} = \delta(t-a)$$

The two shifting theorems

First shifting theorem Let $a \in \mathbb{R}$

and assume that $\mathcal{L}\{f(t)\} = F(s)$

Then:

$$\boxed{\mathcal{L}\left\{e^{at}f(t)\right\} = F(s-a) \quad \text{or} \quad \mathcal{L}^{-1}\left\{F(s-a)\right\} = e^{at}f(t)}$$

Ex/ $\mathcal{L}\left\{e^{-3t}\sin(2t)\right\}$

Sol $\mathcal{L}\left\{\sin(2t)\right\} = \dfrac{2}{s^2+4}$

By First shifting theorem $\mathcal{L}\left\{e^{-3t}\sin(2t)\right\} = F(s-(-3))$

$= F(s+3) = \dfrac{2}{(s+3)^2+4} = \dfrac{2}{s^2+6s+13}$

Ex/ $\mathcal{L}\left\{e^{2t}\dfrac{t^3}{3}\right\}$

Sol we know that $F(s) = \frac{1}{3}\mathcal{L}\left\{\frac{t^3}{3}\right\} = \dfrac{1}{3}\,\dfrac{3!}{s^4} = \dfrac{2}{s^4}$

By the First shifting theorem: $\mathcal{L}\left\{e^{2t}\dfrac{t^3}{3}\right\} = \dfrac{2}{(s-2)^4}$

Ex/ $\mathcal{L}\left\{e^{-4t}\,u(t-\pi)\right\}$

Sol $F(s) = \mathcal{L}\left\{u(t-\pi)\right\} = \dfrac{e^{-\pi s}}{s}$ so $\mathcal{L}\left\{e^{-4t}u(t-\pi)\right\} = \dfrac{e^{-\pi(s+4)}}{(s+4)}$

Ex/ Find $\mathcal{L}^{-1}\left\{\dfrac{3}{(s-5)^6}\right\}$

Ex/ Find $\mathcal{L}\left\{\frac{-}{(s-5)^6}\right\}$

Sol $\mathcal{L}^{-1}\left\{\frac{3}{(s-5)^6}\right\} = \frac{3}{5!}\mathcal{L}^{-1}\left\{\frac{5!}{(s-5)^6}\right\}$

$= \frac{3}{5!} e^{5t} t^5$

Ex/ Find $\mathcal{L}^{-1}\left\{\frac{1}{s^2+2s+2}\right\} = \mathcal{L}^{-1}\left\{\frac{1}{s^2+2s+1-1+2}\right\}$

$= \mathcal{L}^{-1}\left\{\frac{1}{(s+1)^2+1}\right\} = e^{-t}\sin t$

First shifting theorem $F(s) = \mathcal{L}\{f(t)\}$

$\mathcal{L}\{e^{at}f(t)\} = F(s-a) \iff \mathcal{L}^{-1}\{F(s-a)\} = e^{at}f(t)$

Ex/ $\mathcal{L}^{-1}\left\{\frac{s+2}{(s-1)^2(s+3)}\right\}$

Sol $\frac{s+2}{(s-1)^2(s+3)} = \frac{A}{s-1} + \frac{B}{(s-1)^2} + \frac{C}{s+3}$

$\frac{A(s-1)(s+3) + B(s+3) + C(s-1)^2}{(s-1)^2(s+3)}$

$A + C = 0$

$2A + B - 2C = 1$

$-3A + 3B + C = 2$

$\left[\begin{array}{ccc|c} 1 & 0 & 1 & 0 \\ 2 & 1 & -2 & 1 \end{array}\right] \overset{\text{Do stuff}}{\sim} \left[\begin{array}{ccc|c} 1 & 0 & 0 & 1/16 \\ 0 & 1 & 0 & 3/4 \end{array}\right]$

$$\left(\begin{array}{ccc|c} 1 & 0 & 1 & 0 \\ 2 & 1 & -2 & 1 \\ -3 & 3 & 1 & 2 \end{array}\right) \quad \text{Do stuff} \sim \left[\begin{array}{ccc|c} 1 & 0 & 0 & 1/16 \\ 0 & 1 & 0 & 3/4 \\ 0 & 0 & 1 & -1/16 \end{array}\right]$$

$A = 1/16 \qquad B = \frac{3}{4} \qquad C = -\frac{1}{16}$

$$\frac{s+2}{(s-1)^2(s+3)} = \frac{1/16}{s-1} + \frac{3/4}{(s-1)^2} + \frac{1/16}{s+3}$$

$$\mathcal{L}^{-1}\left\{\frac{s+2}{(s-1)^2(s+3)}\right\} = \frac{1}{16}\mathcal{L}^{-1}\left\{\frac{1}{s-1}\right\} + \frac{3}{4}\mathcal{L}^{-1}\left\{\frac{1}{(s-1)^2}\right\} - \frac{1}{16}\mathcal{L}^{-1}\left\{\frac{1}{s+3}\right\}$$

$$= \frac{1}{16}e^{t} + \frac{3}{4}te^{t} - \frac{1}{16}e^{-3t}$$