Last time

Laplace Transform

A function $f(t)$ is called:

① of **Exponential order** α $(\alpha \in \mathbb{R})$ if there exists $M>0$, $t_0>0$ such that

$$|f(t)| \le M e^{\alpha t} \text{ for every } t \ge t_0$$

② Piecewise continuous on $[0, L]$ if there exist a finite subdivision

$$t_0 = 0, t_1, t_2, \dots, t_n = L \text{ of } [0, L]$$

such that

(i) f is continuous on every $]x_i, x_{i+1}[$

(ii) For every i: $\lim_{x \to x_i^-} f(x)$ and $\lim_{x \to x_i^+} f(x)$ exists (are finite numbers)

Improper integral

$$\int_a^{+\infty} f(t)\,dt = \lim_{L \to +\infty} \int_a^L f(t)\,dt$$

Ex/ Determine if the integral

$$\int_0^{+\infty} \frac{1}{1+x^2}\,dx \text{ converges!}$$

Sol $$\int_0^{+\infty} \frac{1}{1+x^2}\,dx = \lim_{L\to+\infty} \int_0^L \frac{1}{1+x^2}\,dx = \lim_{L\to+\infty} \left[\tan^{-1}(x)\right]_0^L$$

$$= \lim_{L\to\infty} \tan^{-1}(L) - \tan^{-1}(0) = \lim \tan^{-1}(L) = \pi/2 < +\infty$$

$\int_0^{+\infty} \frac{1}{1+x^2}\,dx$ converges and its value is $\pi/2$

Theorem: let $f(t)$ be a function of exponential order α and piecewise continuous on $[0, L]$ for every $L > 0$

Then $\int_0^{+\infty} e^{-st} f(t)\,dt$ converges for every $s > \alpha$

Definition Let $f(t)$ be a function of one variable $t \geq 0$ such that:

(i) $f(t)$ is of exponential order α

(ii) $f(t)$ is piecewise continuous on $[0, L]$ where $L > 0$

We difine the Laplace transform of $f(t)$ as the function $F(s)$ as follows:

$$\boxed{F(s) = \mathcal{L}\{f(t)\} = \int_0^{+\infty} e^{-st} f(t)\,dt}\ , \quad s > \alpha$$

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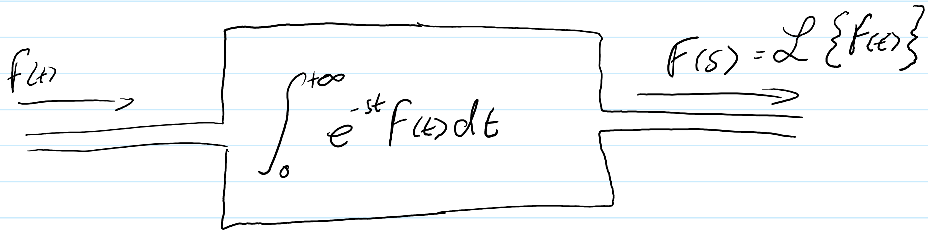

If $F(s) = \mathcal{L}\{f(t)\}$, we write $f(t) = \mathcal{L}^{-1}\{F(s)\}$

$$F(s) = \mathcal{L}\{f(t)\} \Longleftrightarrow f(t) = \mathcal{L}^{-1}\{F(s)\}$$

Theorem $\mathcal{L}$ and $\mathcal{L}^{-1}$ are linear:

1. $\mathcal{L}\{a f(t) + b\, g(t)\} = a\mathcal{L}\{f(t)\} + b\,\mathcal{L}\{g(t)\}$, $a, b, \in \mathbb{C}$

2. $\mathcal{L}^{-1}\{a F(s) + b\, G(s)\} = a\mathcal{L}^{-1}\{F(s)\} + b\,\mathcal{L}^{-1}\{G(s)\}$

$$\mathcal{L}\{f(t)\} = \int_0^{+\infty} e^{-st} f(t)\,dt$$

Laplace transforms of basic functions

1. $f(t) = 1$ Constant Function

$$\mathcal{L}\{1\} = \int_0^{+\infty} e^{-st}(1)\,dt = \lim_{L\to+\infty}\int_0^L e^{-st}(1)\,dt$$

$$= \lim_{L\to+\infty}\left[\frac{-1}{s}e^{-st}\right]_0^L = \lim_{L\to+\infty}\frac{-1}{s}e^{-sL} + \frac{1}{s}$$

$$= \frac{1}{s} \quad , \quad s > 0$$

$$\mathcal{L}\{1\} = \frac{1}{s} \text{ or } \mathcal{L}^{-1}\left\{\frac{1}{s}\right\} = 1$$

② $f(t) = t$

$$\mathcal{L}\{t\} = \int_0^{+\infty} e^{-st}\,t\,dt = \lim_{L\to+\infty}\int_0^L t e^{-st}\,dt$$

By parts: $u = t \qquad v' = e^{-st}$

$u' = 1 \qquad v = \frac{-1}{s}e^{-st}$

$$\int t e^{-st}\,dt = \frac{-t}{s}e^{-st} + \frac{1}{s}\underbrace{\int e^{-st}\,dt}_{\frac{-1}{s}e^{-st}}$$

$$= \frac{-t}{s}e^{-st} - \frac{1}{s^2}e^{-st}$$

$$\mathcal{L}\{t\} = \lim_{L\to+\infty}\left[\frac{-t}{s}e^{-st} - \frac{1}{s^2}e^{-st}\right]_0^L$$

$$= \lim_{L\to\infty}\frac{-L}{s}e^{-sL} - \frac{1}{s^2}e^{-sL} - \left(-\frac{1}{s^2}\right)$$

$= \frac{1}{s^2}$

$\mathcal{L}\{t\} = \frac{1}{s^2}$ or $\mathcal{L}^{-1}\{\frac{1}{s^2}\} = t$, $s > 0$

③ $f(t) = t^n$ In general, we can show that $\mathcal{L}\{t^n\} = \frac{n!}{s^{n+1}}$ (by induction on n)

$$\mathcal{L}\{t^n\} = \frac{n!}{s^{n+1}} \Longleftrightarrow \mathcal{L}^{-1}\left\{\frac{n!}{s^{n+1}}\right\} = t^n, \quad s > 0$$

Ex/ Find $\mathcal{L}\{\frac{2}{3}t^4 - 3t^3 + t^2 - 3\}$

Sol $\mathcal{L}\{\frac{2}{3}t^4 - 3t^3 + t^2 - 3\}$

$$= \frac{2}{3}\mathcal{L}\{t^4\} - 3\mathcal{L}\{t^3\} + \mathcal{L}\{t^2\} - 3\mathcal{L}\{1\}$$

$$= \frac{2}{3}\left(\frac{4!}{s^5}\right) - 3\left(\frac{3!}{s^4}\right) + \left(\frac{2!}{s^3}\right) - 3\left(\frac{1}{s}\right)$$

$$= \frac{16}{s^5} - \frac{18}{s^4} + \frac{2}{s^3} - \frac{3}{s}$$

$$\mathcal{L}^{-1}\left\{\frac{3}{s^7}\right\} = \frac{3}{6!}\mathcal{L}^{-1}\left\{\frac{6!}{s^7}\right\} = \frac{3}{6!}t^6$$

4) $f(t) = e^{at}, \quad a \in \mathbb{C}$

$$\mathcal{L}\{e^{at}\} = \int_0^{+\infty} e^{-st} e^{at}\, dt = \lim_{L \to +\infty} \int_0^L e^{(a-s)t}\, dt$$

$$= \lim_{L \to +\infty} \left[\frac{1}{(a-s)} e^{(a-s)t}\right]_0^L = \lim_{L \to +\infty} \left[\frac{1}{(a-s)} e^{(a-s)L} - \frac{1}{(a-s)}\right], \quad s > a$$

$$= \frac{-1}{a-s} = \frac{1}{s-a}$$

$$\mathcal{L}\{e^{at}\} = \frac{1}{s-a}, \quad s > a \iff \mathcal{L}^{-1}\left\{\frac{1}{s-a}\right\} = e^{at}$$

EX/ $\mathcal{L}\{2e^{-3t} + 4e^{2t} + 3t^2\}$

$$= 2\mathcal{L}\{e^{-3t}\} + 4\mathcal{L}\{e^{2t}\} + 3\mathcal{L}\{t^2\}$$

$$= \frac{2}{s+3} + \frac{4}{s-2} + \frac{3 \cdot 2!}{s^3}$$

EX/ $\mathcal{L}^{-1}\left\{\frac{2s+3}{s^2-5s+6}\right\}$

Sol partial fractions

A B

$$\frac{2s+3}{s^2+5s+6} = \frac{2s+3}{(s-2)(s-3)} = \frac{A}{(s-2)} + \frac{B}{s-3}$$

$$\frac{2s+3}{(s-2)(s-3)} = \frac{As-3A+Bs-2B}{(s-2)(s-3)}$$

$A+B = 2 \quad ① \Rightarrow B = 2-A$

$B = 2+7 = 9$

$-3A-2B = 3 \quad ②$

$-3A-2(2-A) = 3$

$-3A-4+2A = 3$

$-A = 7$

$A = -7$

So $\dfrac{2s+3}{s^2-5s+6} = \dfrac{-7}{(s-2)} + \dfrac{9}{(s-3)}$

$$\mathcal{L}^{-1}\left\{\frac{2s+3}{s^2-5s+6}\right\} = \mathcal{L}^{-1}\left\{\frac{-7}{s-2} + \frac{9}{s-3}\right\}$$

$$= -7\,\mathcal{L}^{-1}\left\{\frac{1}{s-2}\right\} + 9\,\mathcal{L}^{-1}\left\{\frac{1}{s-3}\right\}$$

$$= -7e^{2t} + 9e^{3t}$$

5 $f(t) = \cos(at)$ or $\sin(at)$, $a \in \mathbb{R}$

We know that $\mathcal{L}\left\{e^{iat}\right\} = \dfrac{1}{s-ia}$

$$\mathcal{L}\{\cos(at)+i\sin(at)\} = \frac{s+ia}{(s-ia)(s+ia)}$$

$$\mathcal{L}\{\cos(at)\} + i\mathcal{L}\{\sin(at)\} = \frac{s+ia}{s^2+a^2}$$

$$\underbrace{\mathcal{L}\{\cos(at)\}} + i\underbrace{\mathcal{L}\{\sin(at)\}} = \underbrace{\frac{s}{s^2+a^2}} + i\underbrace{\frac{a}{s^2+a^2}}$$

$\mathcal{L}\{\cos(at)\} = \dfrac{s}{s^2+a^2}$ and $\mathcal{L}\{\sin(at)\} = \dfrac{a}{s^2+a^2}$

$\Updownarrow$ $\Updownarrow$

$\mathcal{L}^{-1}\left\{\dfrac{s}{s^2+a^2}\right\} = \cos(at)$ $\mathcal{L}^{-1}\left\{\dfrac{a}{s^2+a^2}\right\} = \sin(at)$

Ex $\mathcal{L}\{\sin(\sqrt{3}\,t)\} = \dfrac{\sqrt{3}}{s^2+3}$

$$\mathcal{L}\{\cos(\sqrt{3}t)\} = \frac{s}{s^2+3}$$

Ex Find $\mathcal{L}^{-1}\left\{\dfrac{5}{s^2+\underbrace{4}_{a^2}}\right\} = \dfrac{5}{2}\mathcal{L}^{-1}\left\{\dfrac{2}{s^2+4}\right\} = \dfrac{5}{2}\sin(2t)$

Ex Find $\mathcal{L}^{-1}\left\{\dfrac{s}{(s+1)(s^2+1)}\right\}$

Partial Fractions

$$\frac{s}{(s+1)(s^2+1)} = \frac{A}{s+1} + \frac{Bs+C}{s^2+1}$$

$$\frac{s}{(s+1)(s^2+1)} = \frac{As^2 + A + Bs^2 + Bs + Cs + C}{(s+1)(s^2+1)}$$

s^2: $A + B = 0$ ① $\qquad -① + ②: \; -A + C = 1$ ④

s: $B + C = 1$ ② $\qquad ④ + ③: \; 2C = 1 \Rightarrow C = \frac{1}{2}$

$A + C = 0$ ③ $\qquad A = -C = -\frac{1}{2}$

$① - B = \frac{1}{2}$

$$\frac{s}{(s+1)(s^2+1)} = \frac{-\frac{1}{2}}{s+1} + \frac{\frac{1}{2}s + \frac{1}{2}}{s^2+1} = -\frac{1}{2}\frac{1}{s+1} + \frac{1}{2} \cdot \frac{s}{s^2+1} + \frac{1}{2}\frac{1}{s^2+1}$$

$$\mathcal{L}^{-1}\left\{\frac{s}{(s+1)(s^2+1)}\right\} = -\frac{1}{2}\mathcal{L}\left\{\frac{1}{s+1}\right\} + \frac{1}{2}\mathcal{L}^{-1}\left\{\frac{s}{s^2+1}\right\} + \frac{1}{2}\mathcal{L}^{-1}\left\{\frac{1}{s^2+1}\right\}$$

$$\boxed{= -\frac{1}{2}e^{-t} + \frac{1}{2}\cos(t) + \frac{1}{2}\sin(t)}$$