Lec 15

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Non homogeneous systems

$$\vec{y}\,' = A\vec{y} + \vec{f}(x) \qquad (NH)$$

The general solution of (NH) is $\vec{y} = \vec{y}_H + \vec{y}_P$ where

- $\vec{y}_H$ is the solution to the corresponding homogeneous system

$$\vec{y}\,' = A\vec{y} \qquad (H)$$

- $\vec{y}_P$ is one particular solution to (NH)

To find y_P we use the undetermined coefficient method we saw before with one difference:

The word "constant" means "constant vector"

Remark: There is a modification rule in case of systems but we don't consider such a case in this course.

Reminder: if $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$

If $\det(A) \neq 0$, A is invertible and

$$A^{-1} = \frac{1}{\det(A)}\begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$$

Ex/ Solve the equation $A\vec{V} = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$ where

$A = \begin{bmatrix} 1 & -1 \\ 3 & 5 \end{bmatrix}$ for $\vec{V}$

<u>Sol</u>

$$A\vec{V} = \begin{bmatrix} 1 \\ 2 \end{bmatrix} \Rightarrow \vec{V} = A^{-1}\begin{bmatrix} 1 \\ 2 \end{bmatrix}$$

$$\vec{V} = \underbrace{\frac{1}{8}\begin{bmatrix} 5 & 1 \\ -3 & 1 \end{bmatrix}}_{A^{-1}}\begin{bmatrix} 1 \\ 2 \end{bmatrix} = \frac{1}{8}\begin{bmatrix} 7 \\ -1 \end{bmatrix} = \begin{bmatrix} 7/8 \\ -1/8 \end{bmatrix}$$

Ex/ Solve the following IVP:

$$\vec{y}' = \underbrace{\begin{bmatrix} 2 & 0 \\ -1 & -2 \end{bmatrix}}_{A}\vec{y} + \underbrace{\begin{bmatrix} \cos(2x)+1 \\ x+2 \end{bmatrix}}_{f(x)} \quad ; \quad \vec{y}(0) = \begin{bmatrix} 1/4 \\ 1/8 \end{bmatrix}$$

<u>Sol</u> Write the corresponding Homogeneous system is

$$\vec{y}' = A\vec{y} \quad (H)$$

<u>Eigen Values of A</u> $\det(A - \lambda I) = 0 \Rightarrow \det\begin{bmatrix} 2-\lambda & 0 \\ -1 & -2-\lambda \end{bmatrix} = 0$

$\Rightarrow (2-\lambda)(-2-\lambda) = 0 \Rightarrow \lambda_1 = -2 \quad , \lambda_2 = 2$

two distinct real eigenvalues

$\underset{\sim}{E_{-2}} \quad [A - \lambda I \,|\, \vec{0}] = [A + 2I \,|\, \vec{0}]$

$$= \left[\begin{array}{cc|c} 4 & 0 & 0 \\ -1 & 0 & 0 \end{array}\right] \underset{R_1 \leftrightarrow R_2}{\underbrace{-R_2}} \left[\begin{array}{cc|c} 1 & 0 & 0 \\ 4 & 0 & 0 \end{array}\right] \underbrace{-4R_1 + R_2 \to R_2}$$

$$= \left[\begin{array}{cc|c} 1 & 0 & 0 \\ 0 & 0 & 0 \end{array}\right] \qquad \begin{array}{l} x_2 = t \text{ is free} \\ x_1 = 0 \end{array}$$

$$E_{-2} = \left\{ \begin{bmatrix} 0 \\ t \end{bmatrix}, t \in \mathbb{R} \right\} = \left\{ \begin{bmatrix} 0 \\ 1 \end{bmatrix} t \ ; \ t \in \mathbb{R} \right\} = \text{Span} \left\{ \overbrace{\begin{bmatrix} 0 \\ 1 \end{bmatrix}}^{\vec{v}_1} \right\}$$

$E_2 = [A - \lambda_2 I \,|\, \vec{0}] = [A - 2I \,|\, \vec{0}]$

$$= \left[\begin{array}{cc|c} 0 & 0 & 0 \\ -1 & -4 & 0 \end{array}\right] \sim \left[\begin{array}{cc|c} 1 & 4 & 0 \\ 0 & 0 & 0 \end{array}\right] \qquad \begin{array}{l} x_2 = t \\ x_1 = -4t \end{array}$$

$$E_2 = \left\{ \begin{bmatrix} -4t \\ t \end{bmatrix} ; \ t \in \mathbb{R} \right\} = \left\{ \begin{bmatrix} -4 \\ 1 \end{bmatrix} t, \in \mathbb{R} \right\} = \text{Span} \left\{ \overbrace{\begin{bmatrix} -4 \\ 1 \end{bmatrix}}^{\vec{v}_2} \right\}$$

The general solution to (H) is

$$y_H = C_1 \vec{V_1} e^{-2x} + C_2 \vec{V_2} e^{2x}$$

$$= C_1 \begin{bmatrix} 0 \\ 1 \end{bmatrix} e^{-2x} + C_2 \begin{bmatrix} -4 \\ 1 \end{bmatrix} e^{2x} = \begin{bmatrix} -4C_2 e^{2x} \\ C_1 e^{-2x} + C_2 e^{2x} \end{bmatrix}$$

For $\vec{y}_p$ note that

$$\vec{f}_{(x)} = \underbrace{\begin{bmatrix} 1 \\ 0 \end{bmatrix} \cos(2x)} + \overbrace{\begin{bmatrix} 0 \\ 1 \end{bmatrix} x + \begin{bmatrix} 1 \\ 2 \end{bmatrix}}$$

For $\begin{bmatrix} 1 \\ 0 \end{bmatrix} \cos(2x) \longrightarrow \vec{y}_p = \vec{u}\cos(2x) + \vec{V}\sin(2x)$

For $\begin{bmatrix} 0 \\ 1 \end{bmatrix} x + \begin{bmatrix} 1 \\ 2 \end{bmatrix} \longrightarrow \vec{y}_p = \vec{w}x + \vec{T}$

By the sum rule, $\vec{y}_p = \vec{u}\cos(2x) + \vec{V}\sin(2x) + \vec{w}x + \vec{T}$

$\vec{y}_p' = -2\vec{u}\sin(2x) + 2\vec{V}\cos(2x) + \vec{w}$ back to (NH)

$$-2\vec{u}\sin(2x) + 2\vec{V}\cos(2x) + \vec{w} = \underbrace{A\left[\vec{u}\cos(2x) + \vec{V}\sin(2x) + \vec{w}\cancel{x} + \vec{T}\right]} +$$

y_p

$$\begin{bmatrix}1\\0\end{bmatrix}\cos(2x)+\begin{bmatrix}0\\1\end{bmatrix}x+\begin{bmatrix}1\\2\end{bmatrix}$$

$$A\vec{v}=-2\vec{u} \qquad (1)$$

$$A\vec{u}+\begin{bmatrix}1\\0\end{bmatrix}=2\vec{v} \qquad (2)$$

$$A\vec{w}+\begin{bmatrix}0\\1\end{bmatrix}=\begin{bmatrix}0\\0\end{bmatrix} \qquad (3)$$

$$A\vec{T}+\begin{bmatrix}1\\2\end{bmatrix}=\vec{w} \qquad (4)$$

$$(1) \Rightarrow \vec{U}=-\frac{1}{2}\cdot A\vec{V}$$

$$(2) \Rightarrow A\left(-\frac{1}{2}A\vec{V}\right)+\begin{bmatrix}1\\0\end{bmatrix}=2\vec{V} \Rightarrow$$

$$\frac{1}{2}A^2\vec{V}+2\vec{V}=\begin{bmatrix}1\\0\end{bmatrix} \Rightarrow$$

$$\underbrace{\left(\frac{1}{2}A^2+2I\right)}\vec{V}=\begin{bmatrix}1\\0\end{bmatrix} \qquad (*)$$

$$\frac{1}{2}A^2+2I=\frac{1}{2}\begin{bmatrix}2&0\\-1&-2\end{bmatrix}\begin{bmatrix}2&0\\-1&-2\end{bmatrix}+2\begin{bmatrix}1&0\\0&1\end{bmatrix}$$

$$=\frac{1}{2}\begin{bmatrix}4&0\\0&4\end{bmatrix}+\begin{bmatrix}2&0\\0&2\end{bmatrix}=\begin{bmatrix}4&0\\0&4\end{bmatrix}$$

So $(*)$ becomes

$$\begin{bmatrix}4&0\\0&4\end{bmatrix}\vec{V}=\begin{bmatrix}1\\0\end{bmatrix} \Rightarrow \vec{V}=\begin{bmatrix}4&0\\0&4\end{bmatrix}\quad\begin{bmatrix}1\\0\end{bmatrix}$$

$$\begin{bmatrix} 4 & 0 \\ 0 & 4 \end{bmatrix} V = \begin{bmatrix} 1 \\ 0 \end{bmatrix} \Rightarrow \vec{V} = \begin{bmatrix} 4 & 0 \\ 0 & 4 \end{bmatrix} \begin{bmatrix} 1 \\ 0 \end{bmatrix}$$

$$\vec{V} = \frac{1}{16} \begin{bmatrix} 4 & 0 \\ 0 & 4 \end{bmatrix} \begin{bmatrix} 1 \\ 0 \end{bmatrix} = \frac{1}{16} \begin{bmatrix} 4 \\ 0 \end{bmatrix} = \begin{bmatrix} 1/4 \\ 0 \end{bmatrix}$$

$$\vec{V} = \begin{bmatrix} 1/4 \\ 0 \end{bmatrix}$$

From 1, $\vec{u} = -\frac{1}{2} A \vec{V}$

$$\vec{u} = -\frac{1}{2} \begin{bmatrix} 2 & 0 \\ -1 & -2 \end{bmatrix} \begin{bmatrix} 1/4 \\ 0 \end{bmatrix} = -\frac{1}{2} \begin{bmatrix} 1/2 \\ -1/4 \end{bmatrix}$$

$$\vec{u} = \begin{bmatrix} -1/4 \\ 1/8 \end{bmatrix}$$

3 $\Rightarrow A\vec{w} = \begin{bmatrix} 0 \\ -1 \end{bmatrix}$

$$\Rightarrow \vec{w} = A^{-} \begin{bmatrix} 0 \\ -1 \end{bmatrix} = -\frac{1}{4} \begin{bmatrix} -2 & 0 \\ 1 & 2 \end{bmatrix} \begin{bmatrix} 0 \\ -1 \end{bmatrix} = -\frac{1}{4} \begin{bmatrix} 0 \\ -2 \end{bmatrix}$$

$$= \begin{bmatrix} 0 \\ 1/2 \end{bmatrix}$$

4 $\Rightarrow A\vec{T} = \vec{w} - \begin{bmatrix} 1 \\ 2 \end{bmatrix}$

$$A\vec{T} = \begin{bmatrix} 0 \\ 1/2 \end{bmatrix} - \begin{bmatrix} 1 \\ 2 \end{bmatrix} = \begin{bmatrix} -1 \\ -3/2 \end{bmatrix}$$

$$\vec{T} = A^{-1} \begin{bmatrix} -1 \\ -3/2 \end{bmatrix} = -\frac{1}{4} \begin{bmatrix} -2 & 0 \\ 1 & 2 \end{bmatrix} \begin{bmatrix} -1 \\ -3/2 \end{bmatrix} = -\frac{1}{4} \begin{bmatrix} 2 \\ -4 \end{bmatrix} = \begin{bmatrix} -1/2 \\ 1 \end{bmatrix}$$

$$\vec{T} = A^{-1}\begin{bmatrix} -1 \\ -3/2 \end{bmatrix} = -\frac{1}{4}\begin{bmatrix} -2 & 0 \\ 1 & 2 \end{bmatrix}\begin{bmatrix} \ \end{bmatrix} \quad \frac{1}{4}\begin{bmatrix} 2 \\ -4 \end{bmatrix} \quad \begin{bmatrix} -1/2 \\ 1 \end{bmatrix}$$

So the particular solution is: $(\vec{y}_p = \vec{u}\cos(2x) + \vec{v}\sin(2x) + \vec{w}x + \vec{T})$

$$\vec{y}_p = \begin{bmatrix} -1/4 \\ 1/8 \end{bmatrix}\cos(2x) + \begin{bmatrix} 1/4 \\ 0 \end{bmatrix}\sin(2x) + \begin{bmatrix} 0 \\ 1/2 \end{bmatrix}x + \begin{bmatrix} -1/2 \\ 1 \end{bmatrix}$$

$$= \begin{bmatrix} -\frac{1}{4}\cos(2x) + \frac{1}{4}\sin(2x) - \frac{1}{2} \\ \frac{1}{8}\cos(2x) + \frac{1}{2}x + 1 \end{bmatrix}$$

The General solution to (NH) is $\vec{y} = \vec{y}_H + \vec{y}_p$

$$\vec{y} = \begin{bmatrix} -4C_2e^{2x} \\ C_1e^{-2x} + C_2e^{2x} \end{bmatrix} + \begin{bmatrix} -\frac{1}{4}\cos(2x) + \frac{1}{4}\sin(2x) - \frac{1}{2} \\ \frac{1}{8}\cos(2x) + \frac{1}{2}x + 1 \end{bmatrix}$$

$$\vec{y} = \begin{bmatrix} -4C_2e^{2x} - \frac{1}{4}\cos(2x) + \frac{1}{4}\sin(2x) - \frac{1}{2} \\ C_1e^{-2x} + C_2e^{2x} + \frac{1}{8}\cos(2x) + \frac{1}{2}x + 1 \end{bmatrix}$$

$$\vec{y}(0) = \begin{bmatrix} 1/4 \\ 1/8 \end{bmatrix}$$

$$\begin{bmatrix} -4c_2 - \frac{1}{4} - \frac{1}{2} \\ c_1 + c_2 + \frac{1}{8} + 1 \end{bmatrix} = \begin{bmatrix} 1/4 \\ 1/8 \end{bmatrix}$$

$$\begin{cases} -4c_2 - \frac{1}{4} - 1/2 = 1/4 \\ c_1 + c_2 + 1/8 + 1 = 1/8 \end{cases} \Rightarrow \begin{cases} -4c_2 = 1 \\ c_1 + c_2 = -1 \end{cases}$$

So $c_2 = -1/4$

$c_1 = -1 + 1/4 = -3/4$ The solution to the IVP is

$$\vec{y} = \begin{bmatrix} e^{2x} - \frac{1}{4}\cos(2x) + \frac{1}{4}\sin(2x) - 1/2 \\ -3/4\, e^{-2x} - \frac{1}{4} e^{2x} + \frac{1}{8}\cos(2x) + 1/2\, x + 1 \end{bmatrix}$$